VECTOR CALCULUS AND NUMERICAL TECHNIQUES

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VECTOR CALCULUS.

Vector Differentiation.

Differentiation of a Vector Function :-

Let S be a set of real numbers. Corresponding to each scalar $t \in S$, let there be associated a unique vector $\bar{f}$. Then $\bar{f}$ is said to be a vector (vector valued) function. S is called the domain of $\bar{f}$.

We write $\bar{f} = \bar{f}(t)$.

Let $\bar{i}, \bar{j}, \bar{k}$ be three mutually perpendicular unit vectors in three dimensional space. We can write $\bar{f} = \bar{f}(t) = f_1(t)\bar{i} + f_2(t)\bar{j} + f_3(t)\bar{k}$ Where $f_1(t)$ $f_2(t)$ $f_3(t)$ are real valued functions (which are called components of $\bar{f}$)

Derivative :-

Let $\bar{f}$ be a vector function on an interval I and $a \in I$. Then $\underset{t \to a}{Lt} \dfrac{\bar{f}(t) - \bar{f}(a)}{t-a}$, if exists, is called the derivative of $\bar{f}$ at a and is denoted by $\bar{f}'(a)$ (or) $\left(\dfrac{d\bar{f}}{dt}\right)$ at $t=a$. We also say that $\bar{f}$ is differentiable at $t=a$. If $\bar{f}'(a)$ exists.

Higher Order Derivatives :-

Let $\bar{f}$ be differentiable on an interval I and $\bar{f}' = \dfrac{d\bar{f}}{dt}$ be the derivative of $\bar{f}$. If $\underset{t \to a}{Lt} \dfrac{\bar{f}'(t) - \bar{f}'(a)}{t-a}$ exists for every $a \in I_1 \subset I$ then $\bar{f}'$ is said to be differentiable on I_1. It is denoted by $\bar{f}''(a)$ (or) $\dfrac{d^2\bar{f}}{dt^2}$.

Similarly we can define $\bar{f}'''(t)$ etc.

Properties :-

(i) Derivative of a constant vector is $\bar{0}$

(ii) The necessary and sufficient condition for $\bar{f}(t)$ to be constant vector function is $\dfrac{d\bar{f}}{dt} = \bar{0}$

(iii) If $\bar{a}$ and $\bar{b}$ are differentiable vector functions, then.

(a) $\frac{d}{dt}(\bar{a} \pm \bar{b}) = \frac{d\bar{a}}{dt} \pm \frac{d\bar{b}}{dt}$

(b) $\frac{d}{dt}(\bar{a}\cdot\bar{b}) = \frac{d\bar{a}}{dt}\cdot\bar{b} + \bar{a}\cdot\frac{d\bar{b}}{dt}$.

(c) $\frac{d}{dt}(\bar{a}\times\bar{b}) = \frac{d\bar{a}}{dt}\times\bar{b} + \bar{a}\times\frac{d\bar{b}}{dt}$.

(iv) If $\bar{f}$ is a differentiable vector function and ϕ is a scalar differential function, then $\frac{d}{dt}(\phi\bar{f}) = \phi\frac{d\bar{f}}{dt} + \frac{d\phi}{dt}\bar{f}$.

(v) If $\bar{f} = f_1(t)i + f_2(t)j + f_3(t)k$ then $\frac{d\bar{f}}{dt} = \frac{df_1}{dt}i + \frac{df_2}{dt}j + \frac{df_3}{dt}k$. Where $f_1(t), f_2(t), f_3(t)$ are cartesian components of the vector $\bar{f}$.

Partial Derivatives :—

Let $\bar{f}$ be a vector function of scalar variables p, q, t. Then we write $\bar{f} = \bar{f}(p,q,t)$. Treating t as a variable and p, q as constants, we define. $\underset{\delta t \to 0}{Lt}\ \frac{\bar{f}(p,q,t+\delta t) - \bar{f}(p,q,t)}{\delta t}$ if exists, as partial derivative of $\bar{f}$ w.r.t."t" and is denoted by $\frac{\partial\bar{f}}{\partial t}$.

Similarly we can define $\frac{\partial\bar{f}}{\partial p}, \frac{\partial\bar{f}}{\partial q}$ also ..

Properties :—

(i) $\frac{\partial}{\partial t}(\phi\bar{a}) = \frac{\partial\phi}{\partial t}\bar{a} + \phi\frac{\partial\bar{a}}{\partial t}$.

(ii) If λ is a constant, then $\frac{\partial}{\partial t}(\lambda\bar{a}) = \lambda\frac{\partial\bar{a}}{\partial t}$.

(iii) If $\bar{c}$ is a constant vector, then $\frac{\partial}{\partial t}(\phi\bar{c}) = \bar{c}\frac{\partial\phi}{\partial t}$.

(iv) $\frac{\partial}{\partial t}(\bar{a} \pm \bar{b}) = \frac{\partial\bar{a}}{\partial t} \pm \frac{\partial\bar{b}}{\partial t}$.

(v) $\frac{\partial}{\partial t}(\bar{a}\cdot\bar{b}) = \frac{\partial\bar{a}}{\partial t}\cdot\bar{b} + \bar{a}\cdot\frac{\partial\bar{b}}{\partial t}$.

(vi) $\frac{\partial}{\partial t}(\bar{a}\times\bar{b}) = \frac{\partial\bar{a}}{\partial t}\times\bar{b} + \bar{a}\times\frac{\partial\bar{b}}{\partial t}$

(vii) Let $\bar{f} = f_1 i + f_2 j + f_3 k$ where f_1, f_2, f_3 are differentiable scalar functions of more than one variable. Then $\frac{\partial\bar{f}}{\partial t} = i\frac{\partial f_1}{\partial t} + j\frac{\partial f_2}{\partial t} + k\frac{\partial f_3}{\partial t}$.

Vector Point function and vector field :-

Let P be any point in a region 'D' of space. Let r be the position vector of P. If there exists a vector function F corresponding to each P, then such a function F is called a vector point function and the region D is called a vector field.

Eg :- Consider the vector function. $\bar{F} = (x-y)i + xyj + yzk$ —— ①.

Let P be a point whose position vector is $r = 2i + j + 3k$ in the region D of space.

At P, the value of F is obtained by putting $x=2$, $y=1$, $z=3$ in $\bar{F}$

i.e At P, $F = i + 2j + 3k$.

Thus, to each point P of the region D, there corresponds a vector F given by the vector function ①.

Hence F is a vector point function (of scalar variables, x, y, z) and the region D is a vector field.

Eg :- Consider a particle moving in space. At each point P on its path, the particle will be having a velocity $\bar{v}$ which is a vector point function. Similarly, the acceleration of the particle is also a vector point function.

Eg :- In a magnetic field at any point $P(x,y,z)$, there will be a magnetic force $\bar{F}(x,y,z)$. This is called magnetic force field.

Scalar point function and scalar field :-

If there exists a scalar f given by a scalar function f correspon-ding to each point P (with position vector r) in a region D of space. f is called a scalar point function and D is called a scalar field.

Eg:- Let P be a point whose position vector is $r = 2i + j + 3k$.

Consider $f = xyz + xy + z$.

Then the value of f at P is obtained by putting $x=2$, $y=1$, $z=3$.

i.e At P, $f = 2.1.3 + 2.1 + 3 = 11$.

Hence the scalar 11 is attached to the point P.

The function f is a scalar point function (of scalar variables, x, y, z) and D is a scalar field.

Eg:- Consider a heated solid. At each point $P(x,y,z)$ of the solid, there will be tempazature $T(x,y,z)$. This T is a scalar point function.

Eg:- Suppose a particle (or a very small insect) is tracing a path in space. When it occupies a position $P(x,y,z)$ in space, it will be having some speed say, v. This speed v is a scalar point function.

Vector Differential Operator :-

The vector differential operator ∇ (read as del) is defined as

$\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$. ($i, j, k$ are unit vectors in x, y, z directions)

→ This operator ∇ is used in defining the gradient, divergence and curl.

→ Properties of ∇ are similar to those of vectors. The operator is applied to both vector and scalar functions.

Gradient of a scalar point function :-

Let $\phi(x, y, z)$ be a scalar point function of position defined in some region of space. Then the vector function $i\frac{\partial \phi}{\partial x} + j\frac{\partial \phi}{\partial y} + k\frac{\partial \phi}{\partial z}$ is known as the gradient of ϕ and is denoted by grad ϕ or $\nabla \phi$.

$$\nabla \phi = i\frac{\partial \phi}{\partial x} + j\frac{\partial \phi}{\partial y} + k\frac{\partial \phi}{\partial z}.$$

Properties :-

(i) If f and g are two scalar functions then grad $(f \pm g)$ = grad $f \pm$ grad g.

(ii) grad $(fg) = f$(grad g) $+ g$(grad f).

(iii) If c is a constant, grad $(cf) = c$(grad f)

(iv) $\text{grad}\left(\frac{f}{g}\right) = \frac{g(\text{grad} f) - f(\text{grad} g)}{g^2}, (g \neq 0)$.

(v) Let $\bar{r} = xi + yj + zk$ Then $d\bar{r} = dx\,i + dy\,j + dz\,k$. If ϕ is any scalar point function, then $d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz$

$$d\phi = \left(i\frac{\partial \phi}{\partial x} + j\frac{\partial \phi}{\partial y} + k\frac{\partial \phi}{\partial z}\right) \cdot (i\,dx + j\,dy + k\,dz)$$

$$d\phi = \nabla \phi \cdot d\bar{r}$$

(vi) The necessary and sufficient condition for a scalar point function to be constant is that $\nabla \phi = \bar{0}$

(vii) If ϕ defines a scalar field 'grad ϕ' or '$\nabla \phi$' defines a vector field.

Physical significance of grad ϕ :—

If $\phi(x,y,z) = c$ (c being a constant) represents a surface, then 'gradϕ' represents the normal vector to the surface at the point (x,y,z).

For, if $r = xi + yj + zk$ is the position vector of the point (x,y,z) on the surface, we have $dr = dx\, i + dy\, j + dz\, k$ which is in the tangent plane to the surface of $\phi(x,y,z)$.

Again $\nabla\phi \cdot dr = \left[\frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j + \frac{\partial\phi}{\partial z} k\right] \cdot (dx\, i + dy\, j + dz\, k)$

$$= \frac{\partial\phi}{\partial x} dx + \frac{\partial\phi}{\partial y} dy + \frac{\partial\phi}{\partial z} dz .$$

$$= d\phi \quad [\because \phi = c].$$

$\therefore$ The vector $\nabla\phi$ which is perpendicular to the tangent plane is the normal vector to $\phi = c$ at (x,y,z).

Note :- If $\phi(x,y,z) = c$ represents a surface.

(i) Normal to the surface ϕ is $\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$.

(ii) Unit normal vector to the surface ϕ is given by.

$$\bar{n} = \frac{\nabla\phi}{|\nabla\phi|}$$

1) Find the unit normal vector to the surface $z = x^2+y^2$ at the point $(1,-2,5)$

Sol:- Let the surface be $\phi = x^2+y^2-z$.

Given that the point $P(1,-2,5)$,

Normal vector to the surface ϕ is $\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$.

$\frac{\partial\phi}{\partial x} = 2x \quad \frac{\partial\phi}{\partial y} = 2y \quad \frac{\partial\phi}{\partial z} = -1$.

At the point $P(1,-2,5)$

$\frac{\partial\phi}{\partial x} = 2 \quad \frac{\partial\phi}{\partial y} = -4 \quad \frac{\partial\phi}{\partial z} = -1$;

$\therefore \nabla\phi = 2i - 4j - k$

Unit normal vector to the surface ϕ is given by.

$$\bar{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2i-4j-k}{\sqrt{2^2+(-4)^2+(-1)^2}} = \frac{2i-4j-k}{\sqrt{21}}$$

2) Find a unit normal vector to the surface $x^2yz + 4xz^2$ at $(1,-2,-1)$.

Sol:- Let the surface be $\phi = x^2yz + 4xz^2$.

Given that the point $P(1,-2,-1)$

Normal vector to the surface ϕ is $\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$

$\frac{\partial\phi}{\partial x} = 2xyz + 4z^2 \quad \frac{\partial\phi}{\partial y} = x^2z \quad \frac{\partial\phi}{\partial z} = x^2y + 8xz$

At the point $P(1,-2,-1)$ $\quad \frac{\partial\phi}{\partial x} = 8, \frac{\partial\phi}{\partial y} = -1 \quad \frac{\partial\phi}{\partial z} = -10$

$\therefore \ \nabla\phi = 8i - j - 10k.$

Unit normal vector to the surface ϕ is given by

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{8i - j - 10k}{\sqrt{64+1+100}} = \frac{8i - j - 10k}{\sqrt{165}}$$

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If $f = x+y+z$, $g = x^2+y^2+z^2$, $h = xy+yz+zx$. prove that $[\text{grad} f \ \text{grad} g \ \text{grad} h] = 0$

sol: Given that $f = x+y+z$ $\quad g = x^2+y^2+z^2$ $\quad h = xy+yz+zx$.

$$\text{grad} f = \frac{\partial f}{\partial x} i + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}$$

$$\frac{\partial f}{\partial x} = 1 \quad \frac{\partial f}{\partial y} = 1 \quad \frac{\partial f}{\partial z} = 1$$

$$\therefore \text{grad} f = i+j+k.$$

$$\text{grad} g = i\frac{\partial g}{\partial x} + j\frac{\partial g}{\partial y} + k\frac{\partial g}{\partial z}$$

$$\frac{\partial g}{\partial x} = 2x \quad \frac{\partial g}{\partial y} = 2y \quad \frac{\partial g}{\partial z} = 2z$$

$$\text{grad} g = 2x i + 2y j + 2z k.$$

$$\text{grad} h = i\frac{\partial h}{\partial x} + j\frac{\partial h}{\partial y} + k\frac{\partial h}{\partial z}.$$

$$\frac{\partial h}{\partial x} = y+z \quad \frac{\partial h}{\partial y} = x+z \quad \frac{\partial h}{\partial z} = x+y$$

$$\text{grad} h = (y+z)i + (x+z)j + (x+y)k.$$

$$[\text{grad} f \ \text{grad} g \ \text{grad} h] = \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ y+z & x+z & x+y \end{vmatrix}$$

$$C_2 \to C_2 - C_1 \quad C_3 \to C_3 - C_1$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ 2x & 2(y-x) & 2(z-x) \\ y+z & -y+x & x-z \end{vmatrix}$$

$$= (y-x)(z-x)\begin{vmatrix} 1 & 0 & 0 \\ 2x & 2 & 2 \\ y+z & -1 & -1 \end{vmatrix} = 0$$

$$[\text{grad} f \ \text{grad} g \ \text{grad} h] = 0.$$

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Angle between two surfaces :-

Let $\phi_1(x,y,z)$ and $\phi_2(x,y,z)$ be two surfaces.

We know that the angle between two surfaces at a point is the angle between the normals to the surfaces at that point.

Normal vector to the surface ϕ_1 is given by

$$\nabla\phi_1 = i\frac{\partial\phi_1}{\partial x} + j\frac{\partial\phi_1}{\partial y} + k\frac{\partial\phi_1}{\partial z}.$$

Normal vector to the surface ϕ_2 is given by

$$\nabla\phi_2 = i\frac{\partial\phi_2}{\partial x} + j\frac{\partial\phi_2}{\partial y} + k\frac{\partial\phi_2}{\partial z}.$$

Let $\bar{a} = \nabla\phi_1$ $\quad \bar{b} = \nabla\phi_2$.

Let θ be the angle between the two surfaces Then.

$$\cos\theta = \frac{\bar{a}\cdot\bar{b}}{|\bar{a}||\bar{b}|}.$$

Note :-

(i) If $\theta = \frac{\pi}{2}$, $\quad \bar{a}\cdot\bar{b} = 0$ $\quad$ Then two surfaces ϕ_1 and ϕ_2 are $\perp$ers.

(ii) If $\theta = 0$, $\quad \bar{a}\cdot\bar{b} = |\bar{a}||\bar{b}|$.

Find the acute angle between the surface $xy^2z = 2$ and $x^2+y^2+z^2=6$ at the point $(2,1,1)$.

Sol:- Let $\phi_1 = xy^2z - 2$ and $\phi_2 = x^2+y^2+z^2-6$.

We know that the angle between two surfaces at a point is the angle between the normals to the surfaces at that point.

Normal to the surface ϕ_1 is $\nabla\phi_1 = i\frac{\partial\phi_1}{\partial x} + j\frac{\partial\phi_1}{\partial y} + k\frac{\partial\phi_1}{\partial z}$.

$$\frac{\partial\phi_1}{\partial x} = y^2z \qquad \frac{\partial\phi_1}{\partial y} = 2xyz \qquad \frac{\partial\phi_1}{\partial z} = xy^2$$

At the point $(2,1,1)$ $\quad \frac{\partial\phi_1}{\partial x} = 1 \quad \frac{\partial\phi_1}{\partial y} = 4 \quad \frac{\partial\phi_1}{\partial z} = 2$

$$\bar{a} = \nabla\phi_1 = i+4j+2k.$$

Normal to the surface ϕ_2 is $\nabla\phi_2 = i\frac{\partial\phi_2}{\partial x} + j\frac{\partial\phi_2}{\partial y} + k\frac{\partial\phi_2}{\partial z}$.

$$\frac{\partial\phi_2}{\partial x} = 2x \qquad \frac{\partial\phi_2}{\partial y} = 2y \qquad \frac{\partial\phi_2}{\partial z} = 2z.$$

At the point $(2,1,1)$ $\quad \frac{\partial\phi_2}{\partial x} = 4 \quad \frac{\partial\phi_2}{\partial y} = 2 \quad \frac{\partial\phi_2}{\partial z} = 2$

$$\bar{b} = \nabla\phi_2 = 4i+2j+2k.$$

The vectors $\bar{a}$ and $\bar{b}$ are along the normals to the surfaces ϕ_1 and ϕ_2 at the pt $(2,1,1)$.

Let θ be the angle between two surfaces. Then

$$\cos\theta = \frac{\bar{a}\cdot\bar{b}}{|\bar{a}|\,|\bar{b}|}$$

$$\cos\theta = \left|\frac{(i+4j+2k)\cdot(4i+2j+2k)}{\sqrt{1+4^2+2^2}\,\sqrt{4^2+2^2+2^2}}\right| = \frac{4+8+4}{\sqrt{1+16+4}\,\sqrt{16+4+4}}$$

$$\cos\theta = \frac{16}{\sqrt{21}\,\sqrt{24}}$$

$$\theta = \cos^{-1}\left(\frac{8}{3\sqrt{14}}\right).$$

Find the angle between the normals to the surface $xy = z^2$ at the points $(4,1,2)$ and $(3,3,-3)$.

Sol:- Given that the surface $\phi = xy - z^2$.

Wkt the normal to the surface ϕ is given by $\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$

$$\frac{\partial\phi}{\partial x} = y \quad \frac{\partial\phi}{\partial y} = x \quad \frac{\partial\phi}{\partial z} = -2z.$$

$$\nabla\phi = yi + xj - 2zk.$$

Let $\bar{a}$ and $\bar{b}$ be the normals to the surface ϕ at $(4,1,2)$ and $(3,3,-3)$

At the pt $(4,1,2)$

$$\bar{a} = i + 4j - 4k.$$

At the pt $(3,3,-3)$

$$\bar{b} = 3i + 3j + 6k.$$

Let θ be the angle between the two normals.

$$\therefore \quad \cos\theta = \frac{\bar{a}\cdot\bar{b}}{|\bar{a}|\,|\bar{b}|}$$

$$\cos\theta = \frac{(i+4j-4k)\cdot(3i+3j+6k)}{\sqrt{1+16+16}\ \sqrt{9+9+36}}$$

$$\cos\theta = \frac{3+12-24}{\sqrt{33}\ \sqrt{54}} = \frac{-9}{\sqrt{33}\ \sqrt{54}}$$

Find the constants p and q such that the surfaces $px^2 - qyz = (p+2)x$ and $4x^2y + z^3 = 4$ are orthogonal at the point $(1,-1,2)$.

Sol: Let $\phi_1 = px^2 - qyz - (p+2)x$ —— ①

$\phi_2 = 4x^2y + z^3 - 4$. —— ②

Given that the surfaces ϕ_1 and ϕ_2 are orthogonal at the point $(1,-1,2)$

Since the point $(1,-1,2)$ lies on ① and ②.

We have $p + 2q - p - 2 = 0$ [$\because$ from ①]

$q = 1$.

Normal to the surface ϕ_1 is given by $\nabla\phi_1 = i\frac{\partial\phi_1}{\partial x} + j\frac{\partial\phi_1}{\partial y} + k\frac{\partial\phi_1}{\partial z}$

$\frac{\partial\phi_1}{\partial x} = 2px - p - 2$ $\quad \frac{\partial\phi_1}{\partial y} = -qz$ $\quad \frac{\partial\phi_1}{\partial z} = -qy$.

$\nabla\phi_1 = (2px - p - 2)i - qzj - qyk$.

At the point $(1,-1,2)$

$\nabla\phi_1 = (p-2)i - 2qj + qk = \bar{a}$ (say)

Normal to the surface ϕ_2 is given by $\nabla\phi_2 = i\frac{\partial\phi_2}{\partial x} + j\frac{\partial\phi_2}{\partial y} + k\frac{\partial\phi_2}{\partial z}$.

$\frac{\partial\phi_2}{\partial x} = 8xy$ $\quad \frac{\partial\phi_2}{\partial y} = 4x^2$ $\quad \frac{\partial\phi_2}{\partial z} = 3z^2$:

$\nabla\phi_2 = (8xy)i + (4x^2)j + (3z^2)k$

At the point $(1,-1,2)$

$\nabla\phi_2 = -8i + 4j + 12k = \bar{b}$ (say)

Angle between the surfaces is given by $\cos\theta = \frac{\bar{a}\cdot\bar{b}}{|\bar{a}|\,|\bar{b}|}$.

Since the surfaces ① and ② are orthogonal (Perpendicular) i.e $\theta = \frac{\pi}{2}$

Then $\bar{a}\cdot\bar{b} = 0$

$[(p-2)i - 2qj + qk]\cdot[-8i + 4j + 12k] = 0$.

$$-8(p-2)+4(-2q)+12q=0$$

$$-8p+16-8q+12q=0$$

$$2p-q=4$$

$$2p-1=4 \quad [\because q=1]$$

$$p=\frac{5}{2}$$

$\therefore$ The values of p and q are $p=\frac{5}{2}$, $q=1$.

Find the angle of intersection of the spheres $x^2+y^2+z^2=29$ and $x^2+y^2+z^2+4x-6y-8z-47=0$ at the point $(4,-3,2)$.

sol:- Let $f=x^2+y^2+z^2-29$ $g=x^2+y^2+z^2+4x-6y-8z-47$.

WKT the angle between two surfaces at a point is the angle between the normals to the surfaces at that point.

Normal to the surface f is given by $\nabla f = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}$

$$\frac{\partial f}{\partial x} = 2x \quad \frac{\partial f}{\partial y} = 2y \quad \frac{\partial f}{\partial z} = 2z$$

At the pt $(4,-3,2)$ = $\frac{\partial f}{\partial x} = 8$ $\frac{\partial f}{\partial y} = -6$ $\frac{\partial f}{\partial z} = 4$

$$\bar{a} = \nabla f = 8i - 6j + 4k.$$

Normal to the surface g is given by $\nabla g = i\frac{\partial g}{\partial x} + j\frac{\partial g}{\partial y} + k\frac{\partial g}{\partial z}$.

$$\frac{\partial g}{\partial x} = 2x+4 \quad \frac{\partial g}{\partial y} = 2y-6 \quad \frac{\partial g}{\partial z} = 2z-8.$$

At the pt $(4,-3,2)$ $\frac{\partial g}{\partial x} = 12$ $\frac{\partial g}{\partial y} = -12$ $\frac{\partial g}{\partial z} = -4$

$$\bar{b} = \nabla g = 12i - 12j - 4k.$$

Let θ be the angle b/n the normals $\bar{a}$ and $\bar{b}$. Then

$$\cos\theta = \frac{\bar{a}\cdot\bar{b}}{|\bar{a}|\,|\bar{b}|}$$

$$\cos\theta = \frac{(8i-6j+4k)\cdot(12i-12j-4k)}{\sqrt{64+36+16}\,\sqrt{144+144+16}}$$

$$\cos\theta = \frac{154}{\sqrt{116}\,\sqrt{304}}$$

$$\theta = \cos^{-1}\left(\sqrt{\frac{19}{29}}\right)$$

Find the angle between the normals to the surface $x^2 = yz$ at the points $(1,1,1)$ and $(2,4,1)$.

Sol:- Let $\phi = x^2 - yz$.

Let $\bar{a}$ and $\bar{b}$ be the normals to this surface at the points $(1,1,1)$ and $(2,4,1)$ respectively.

Normal to the surface ϕ is $\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$.

$$\frac{\partial\phi}{\partial x} = 2x \qquad \frac{\partial\phi}{\partial y} = -z \qquad \frac{\partial\phi}{\partial z} = -y.$$

$$\therefore \quad \nabla\phi = (2x)\,i + (-z)\,j + (-y)\,k.$$

At the point $P_1(1,1,1)$ $\quad \bar{a} = \nabla\phi = 2i - j - k$.

At the point $P_2(2,4,1)$ $\quad \bar{b} = \nabla\phi = 4i - j - 4k$

Let θ be the angle between the normals $\bar{a}$ and $\bar{b}$. Then

$$\cos\theta = \frac{\bar{a}\cdot\bar{b}}{|\bar{a}|\,|\bar{b}|}$$

$$\cos\theta = \frac{(2i-j-k)\cdot(4i-j-4k)}{\sqrt{4+1+1}\,\sqrt{16+16+1}}$$

$$\cos\theta = \frac{8+1+4}{\sqrt{6}\,\sqrt{33}}$$

$$\theta = \cos^{-1}\left(\frac{13}{\sqrt{198}}\right)$$

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Directional Derivative :-

Let $\phi(x,y,z)$ be a scalar function defined throughout some region of space.

If $\bar{a}$ be any vector. $\frac{\nabla\phi\cdot\bar{a}}{|\bar{a}|}$ which represents the component of $\nabla\phi$ in the direction of $\bar{a}$ is known as the directional derivative of ϕ in the direction of $\bar{a}$.

Note :- (i) The Directional Derivative of ϕ in the direction of $\bar{a}$ is given by $\nabla\phi\cdot\frac{\bar{a}}{|\bar{a}|}$

(ii) Physically the directional derivative is the rate of change of ϕ in the direction of $\bar{a}$.

(iii) The directional derivative will be maximum in the direction of $\nabla\phi$ (i.e $\bar{a}=\nabla\phi$) and the maximum value of the directional derivative

$$\nabla\phi\cdot\frac{\bar{a}}{|\bar{a}|} = \nabla\phi\cdot\frac{\nabla\phi}{|\nabla\phi|} = \frac{|\nabla\phi|^2}{|\nabla\phi|} = |\nabla\phi|$$

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TYPE-1

1) Find the directional derivative of $2xy+z^2$ at $(1,-1,3)$ in the direction of $i+2j+3k$. Also find maximum directional derivative.

Sol:- The directional derivative of ϕ in the direction of $\bar{a}$ is given by

$$\nabla\phi \cdot \frac{\bar{a}}{|\bar{a}|}$$

Let $\phi = 2xy+z^2$, $\bar{a} = i+2j+3k$.

WKT $\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$.

$$\frac{\partial\phi}{\partial x} = 2y \qquad \frac{\partial\phi}{\partial y} = 2x \qquad \frac{\partial\phi}{\partial z} = 2z$$

At the point $(1,-1,3)$, $\frac{\partial\phi}{\partial x} = -2$ $\quad \frac{\partial\phi}{\partial y} = 2$ $\quad \frac{\partial\phi}{\partial z} = 6$.

$\therefore \nabla\phi = -2i+2j+6k$.

$\therefore$ Directional Derivative $= \nabla\phi \cdot \frac{\bar{a}}{|\bar{a}|}$

$$= (-2i+2j+6k)\cdot\frac{(i+2j+3k)}{\sqrt{1+4+9}}$$

$$= \frac{-2+4+18}{\sqrt{14}} = \frac{20}{\sqrt{14}}.$$

Maximum Directional Derivative $= |\nabla\phi| = \sqrt{(-2)^2+2^2+6^2}$

$$= \sqrt{4+4+36}$$

$$= \sqrt{44}.$$

2) Find the directional derivative of $\phi = x^2yz+4xz^2$ at $(1,-2,-1)$ in the direction $2i-j-2k$. Also find maximum directional derivative.

Ans:- $\frac{37}{3}$

Sol:- Given that $\phi = x^2yz+4xz^2$.

Let $\bar{a} = 2i-j-2k$

The directional derivative of ϕ in the direction of $\bar{a}$ is given by

$$\nabla\phi \cdot \frac{\bar{a}}{|\bar{a}|}$$

WKT $\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$

$\frac{\partial\phi}{\partial x} = 2xyz + 4z^2 \quad \frac{\partial\phi}{\partial y} = x^2z \quad \frac{\partial\phi}{\partial z} = x^2y + 8xz$

At the point $(1,-2,-1)$ $\quad \frac{\partial\phi}{\partial x} = 8 \quad \frac{\partial\phi}{\partial y} = -1 \quad \frac{\partial\phi}{\partial z} = -10$

$\therefore \nabla\phi = 8i - j - 10k.$

Directional Derivative $= \nabla\phi \cdot \frac{\bar{a}}{|\bar{a}|}$

$= (8i-j-10k) \cdot \frac{(2i-j-2k)}{\sqrt{4+1+4}}$

$= \frac{16+1+20}{3}$

$= \frac{37}{3}.$

Maximum Directional Derivative $= |\nabla\phi| = \sqrt{8^2+(-1)^2+(-10)^2} = \sqrt{165}$

TYPE-2 :

1) Find the directional derivative of the function $x^2-y^2+2z^2$ at the point $P(1,2,3)$ in the direction of the line PQ where $Q(5,0,4)$

Sol:- WKT the directional derivative of ϕ in the direction of $\bar{a}$ is given by $\nabla\phi \cdot \frac{\bar{a}}{|\bar{a}|}$.

Let $\phi = x^2-y^2+2z^2$

$$\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$

$$\frac{\partial\phi}{\partial x} = 2x \qquad \frac{\partial\phi}{\partial y} = -2y \qquad \frac{\partial\phi}{\partial z} = 4z$$

At the point $P(1,2,3)$ $\quad \frac{\partial\phi}{\partial x} = 2, \quad \frac{\partial\phi}{\partial y} = -4 \quad \frac{\partial\phi}{\partial z} = 12$.

$$\therefore \nabla\phi = 2i-4j+12k.$$

The position vectors of P and Q with respect to the origin are

$\overline{OP} = i+2j+3k$ and $\overline{OQ} = 5i+0j+4k$.

$$\therefore \overline{PQ} = \overline{OQ}-\overline{OP} = 4i-2j+k.$$

$$\text{Let } \bar{a} = 4i-2j+k.$$

$$\therefore \text{Directional Derivative} = \nabla\phi \cdot \frac{\bar{a}}{|\bar{a}|}$$

$$= (2i-4j+12k) \cdot \frac{4i-2j+k}{\sqrt{16+4+1}}$$

$$= \frac{8+8+12}{\sqrt{21}}$$

$$= \frac{28}{\sqrt{21}}$$

$$\therefore \text{Maximum Directional Derivative} = |\nabla\phi| = \sqrt{2^2+(-4)^2+12^2}$$

$$= \sqrt{4+16+144}$$

$$= \sqrt{164}$$

2) Find the directional derivative of the scalar point function $4xy^2 + 2x^2yz$ at the point $A(1,2,3)$ in the direction of the line AB where $B(5,0,4)$. Also find maximum directional derivative.

[Ans: $\frac{120}{\sqrt{21}}$]

Sol:- We know that the directional derivative of ϕ in the direction of $\bar{a}$ is given by $\nabla\phi \cdot \frac{\bar{a}}{|\bar{a}|}$.

Let $\phi = 4xy^2 + 2x^2yz$

$$\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$

$$\frac{\partial\phi}{\partial x} = 4y^2 + 4xyz \quad \frac{\partial\phi}{\partial y} = 8xy + 2x^2z, \quad \frac{\partial\phi}{\partial z} = 2x^2y$$

At the point $P(1,2,3)$ $\quad \frac{\partial\phi}{\partial x} = 40 \quad \frac{\partial\phi}{\partial y} = 22 \quad \frac{\partial\phi}{\partial z} = 4$

$$\nabla\phi = 40i + 22j + 4k.$$

The position vectors of P and Q with respect to the origin are

$\overline{OP} = i+2j+3k, \quad \overline{OQ} = 5i+0j+4k.$

$$PQ = \overline{OQ} - \overline{OP} = 4i - 2j + k$$

$$\text{Let } \bar{a} = 4i - 2j + k$$

$\therefore$ Directional Derivative $= \nabla\phi \cdot \frac{\bar{a}}{|\bar{a}|}$

$$= (40i + 22j + 4k) \cdot \frac{(4i - 2j + k)}{\sqrt{16+4+1}}$$

$$= \frac{160 - 44 + 4}{\sqrt{21}}$$

$$= \frac{200}{\sqrt{21}}$$

$\therefore$ Maximum Directional Derivative $= |\nabla\phi| = \sqrt{(40)^2 + (22)^2 + 16}$

$$= \sqrt{1600 + 484 + 16}$$

$$= \sqrt{2100}.$$

Note :- Let the equation of the curve be $x = x_1(t)$ $y = y_1(t)$, $z = z_1(t)$ — ①

Let $\bar{r}$ be the position vector of any point on the curve ①.

Then $\bar{r} = x_1(t)\, i + y_1(t)\, j + z_1(t)\, k$. $\left[\because \bar{r} = xi + yj + zk\right]$.

$\frac{d\bar{r}}{dt}$ is the vector along the tangent to the curve ①.

TYPE-3

(1) Find the directional derivative of the function $xy^2 + yz^2 + zx^2$ along the tangent to the curve $x = t$, $y = t^2$, $z = t^3$ at the point $(1,1,1)$.

Sol:- WKT. the directional derivative of ϕ in the direction of $\bar{a}$ is given by $\nabla\phi \cdot \frac{\bar{a}}{|\bar{a}|}$

Let $\phi = xy^2 + yz^2 + zx^2$

$$\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}.$$

$$\frac{\partial\phi}{\partial x} = (y^2 + 2xz) \qquad \frac{\partial\phi}{\partial y} = 2xy + z^2 \qquad \frac{\partial\phi}{\partial z} = 2yz + x^2$$

At the pt $(1,1,1)$ $\frac{\partial\phi}{\partial x} = 3$ $\frac{\partial\phi}{\partial y} = 3$ $\frac{\partial\phi}{\partial z} = 3$.

$$\therefore \nabla\phi = 3i + 3j + 3k.$$

Given that the curve $x = t$ $y = t^2$ $z = t^3$ — ①

Let $\bar{r}$ be the position vector of any point on the curve ①

$$\bar{r} = xi + yj + zk.$$

$$\bar{r} = ti + t^2 j + \cancel{3}t^3 k$$

$$\frac{d\bar{r}}{dt} = i + 2tj + 3t^2 k$$

At the pt $(1,1,1)$ $\frac{d\bar{r}}{dt} = i + 2j + 3k$

Let $\bar{a} = i + 2j + 3k$.

$x = t$, $y = t^2$, $z = t^3$
We have pt $(1,1,1)$
At $x = 1$, $t = 1$
At $y = 1$, $t = 1$
At $z = 1$, $t = 1$
$\therefore$ The t value is 1
$t = 1$.

$$\text{Directional Derivative} = \nabla\phi \cdot \frac{\bar{a}}{|\bar{a}|}$$

$$= (3i+3j+3k) \cdot \frac{(i+2j+3k)}{\sqrt{1+4+9}} = \frac{3+6+9}{\sqrt{14}}$$

$$= \frac{18}{\sqrt{14}}$$

Maximum Directional Derivative $= |\nabla\phi| = \sqrt{3^2+3^2+3^2} = \sqrt{27} = 3\sqrt{3}$.

2) Find the directional derivative of $x^2y^2+y^2z^2+z^2x^2$ at the point $(1,1,-2)$ in the direction of the tangent to the curve $x = e^{-t}$, $y = 2\sin t + 1$, $z = t - \cos t$ at $t=0$. Also find maximum directional derivative. $\left[\text{Ans:- } \frac{2}{\sqrt{6}}\right]$

Sol:- The directional derivative of ϕ in the direction of $\bar{a}$ is given by $\nabla\phi \cdot \frac{\bar{a}}{|\bar{a}|}$.

Let $\phi = x^2y^2+y^2z^2+z^2x^2$

$$\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}.$$

$$\frac{\partial\phi}{\partial x} = 2xy^2+2xz^2 \quad \frac{\partial\phi}{\partial y} = 2yz^2+2yx^2 \quad \frac{\partial\phi}{\partial z} = 2zy^2+2zx^2$$

At the point $(1,1,-2)$. $\frac{\partial\phi}{\partial x} = 10$ $\quad \frac{\partial\phi}{\partial y} = 10$, $\frac{\partial\phi}{\partial z} = -8$

$$\nabla\phi = 10i+10j-8k.$$

Given that the curve $x = e^{-t}$ $y = 2\sin t+1$ $z = t-\cos t$ —— (1)

Let $\bar{r}$ be the position vector of any point on the curve (1)

$$\bar{r} = xi+yj+zk$$

$$\bar{r} = e^{-t}i + (2\sin t+1)j + (t-\cos t)k$$

$$\frac{d\bar{r}}{dt} = -e^{-t}i + 2\cos t\, j + (1+\sin t)k$$

At $t=0$, $\frac{d\bar{r}}{dt} = -i+2j+k$

Let $\bar{a} = -i+2j+k$.

Directional Derivative $= \nabla\phi \cdot \frac{\bar{a}}{|\bar{a}|}$

$$= (10i + 10j - 8k) \cdot \frac{(-i + 2j + k)}{\sqrt{1+4+1}}$$

$$= \frac{-10 + 20 - 8}{\sqrt{6}}$$

$$= \frac{2}{\sqrt{6}}$$

Maximum Directional Derivative $= |\nabla\phi| = \sqrt{10^2 + 10^2 + (-8)^2}$

$$= \sqrt{264}$$

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TYPE-4 :-

1) Find the directional derivative of xyz^2+xz at $(1,1,1)$ in a direction of the normal to the surface $3xy^2+y=z$ at $(0,1,1)$.

Sol: WKT The directional derivative of ϕ in the direction of $\bar{a}$ is given by $\nabla\phi \cdot \frac{\bar{a}}{|\bar{a}|}$

Let $\phi = xyz^2+xz$

$$\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$

$$\frac{\partial\phi}{\partial x} = yz^2+z \qquad \frac{\partial\phi}{\partial y} = xz^2 . \qquad \frac{\partial\phi}{\partial z} = 2xyz+x$$

At the pt $(1,1,1)$ $\quad \frac{\partial\phi}{\partial x} = 2 \quad \frac{\partial\phi}{\partial y} = 1 \quad \frac{\partial\phi}{\partial z} = 3 .$

$$\nabla\phi = 2i+j+3k .$$

Let $f = 3xy^2+y-z$

Normal to the surface f is given by $\nabla f = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}$

$$\frac{\partial f}{\partial x} = 3y^2 \qquad \frac{\partial f}{\partial y} = 6xy+1 \qquad \frac{\partial f}{\partial z} = -1.$$

At the pt $(0,1,1)$ $\quad \frac{\partial f}{\partial x} = 3 \quad \frac{\partial f}{\partial y} = 1 \quad \frac{\partial f}{\partial z} = -1$

$$\nabla f = 3i+j-k$$

Let $\bar{a} = 3i+j-k$

$\therefore$ Directional Derivative $= \nabla\phi \cdot \frac{\bar{a}}{|\bar{a}|}$

$$= (2i+j+3k)\cdot\frac{(3i+j-k)}{\sqrt{9+1+1}} = \frac{6+1-3}{\sqrt{11}} = \frac{4}{\sqrt{11}}$$

$\therefore$ Maximum Directional Derivative $= |\nabla\phi| = \sqrt{4+1+9} = \sqrt{14}.$

(2) Find the directional derivative of $x^2yz + 4xz^2$ at the point $(1,-2,-1)$ in the direction of the normal to the surface $x\log z - y^2$ at $(-1,2,1)$. Also find maximum directional derivative. Ans:- $\frac{14}{\sqrt{17}}$

Sol:- The directional derivative of ϕ in the direction of $\bar{a}$ is given by

$\nabla\phi \cdot \frac{\bar{a}}{|\bar{a}|}$.

Let $\phi = x^2yz + 4xz^2$

$$\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}.$$

$$\frac{\partial\phi}{\partial x} = 2xyz + 4z^2 \quad \frac{\partial\phi}{\partial y} = x^2z \quad \frac{\partial\phi}{\partial z} = x^2y + 8xz.$$

At the point $(1,-2,-1)$ $\frac{\partial\phi}{\partial x} = 8$ $\frac{\partial\phi}{\partial y} = -1$ $\frac{\partial\phi}{\partial z} = -10$.

$$\nabla\phi = 8i - j - 10k.$$

Let $f = x\log z - y^2$.

Normal to the surface f is given by $\nabla f = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}$

$$\frac{\partial f}{\partial x} = \log z \quad \frac{\partial f}{\partial y} = -2y \quad \frac{\partial f}{\partial z} = \frac{x}{z}$$

At the point $(-1,2,1)$ $\frac{\partial f}{\partial x} = 0$ $\frac{\partial f}{\partial y} = -4$ $\frac{\partial f}{\partial z} = -1$.

$$\nabla f = -4j - k.$$

Let $\bar{a} = -4j - k$.

$\therefore$ Directional Derivative $= \nabla\phi \cdot \frac{\bar{a}}{|\bar{a}|}$

$$= (8i - j - 10k) \cdot \frac{(-4j - k)}{\sqrt{16+1}}$$

$$= \frac{4+10}{\sqrt{17}}$$

$$= \frac{14}{\sqrt{17}}$$

Maximum Directional Derivative $= |\nabla\phi| = \sqrt{8^2 + (-1)^2 + (-10)^2} = \sqrt{165}$.

TYPE-5 :-

1) In what direction from the point $(-1,1,2)$ is the directional derivative of xy^2z^3 maximum. What is the magnitude of the maximum.

Sol: Let $\phi = xy^2z^3$

$$\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$

$$\frac{\partial\phi}{\partial x} = y^2z^3 \quad \frac{\partial\phi}{\partial y} = 2xyz^3 \quad \frac{\partial\phi}{\partial z} = 3xy^2z^2$$

At the pt $(-1,1,2)$ $\quad \frac{\partial\phi}{\partial x} = 8 \quad \frac{\partial\phi}{\partial y} = -16 \quad \frac{\partial\phi}{\partial z} = -12$

$$\nabla\phi = 8i - 16j - 12k.$$

WKT the directional derivative of ϕ is maximum in the direction of $\nabla\phi$.

$\therefore$ The directional derivative is maximum in the direction of $\nabla\phi$.

$\therefore$ The magnitude of this maximum is $|\nabla\phi| = \sqrt{8^2 + (-16)^2 + (-12)^2}$

$= \sqrt{464}$

2) Find the maximum value of the directional derivative of $\phi = x^2yz$ at $(1,4,1)$

[Ans:- 9]

Sol:- Let $\phi = x^2yz$.

$$\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}.$$

$$\frac{\partial\phi}{\partial x} = 2xyz \quad \frac{\partial\phi}{\partial y} = x^2z \quad \frac{\partial\phi}{\partial z} = x^2y.$$

At the point $(1,4,1)$ $\quad \frac{\partial\phi}{\partial x} = 8 \quad \frac{\partial\phi}{\partial y} = 1 \quad \frac{\partial\phi}{\partial z} = 4$

$$\nabla\phi = 8i + j + 4k.$$

WKT the directional derivative of ϕ is maximum in the direction of $\nabla\phi$.

$\therefore$ The directional derivative is maximum in the direction of $\nabla\phi$.

$\therefore$ The magnitude of this maximum is $|\nabla\phi| = \sqrt{8^2 + 1^2 + 4^2}$

$= \sqrt{69}$.

TYPE - 6

→ Find the directional derivative of $5x^2y - 5y^2z + 2.5z^2x$ at the point $P(1,1,1)$ in the direction of the line $\frac{x-1}{2} = \frac{y-3}{-2} = z$.

Sol:- We know that the directional derivative of the function ϕ in the direction of $\bar{a}$ is given by $\nabla\phi \cdot \frac{\bar{a}}{|\bar{a}|}$

Let $\phi = 5x^2y - 5y^2z + 2.5z^2x$

$$\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}.$$

$\frac{\partial\phi}{\partial x} = 10xy + 2.5z^2 \quad \frac{\partial\phi}{\partial y} = -10yz + 5x^2 \quad \frac{\partial\phi}{\partial z} = -5y^2 + 5zx.$

At the point $(1,1,1)$ $\frac{\partial x}{\partial x} = 12.5 \quad \frac{\partial\phi}{\partial y} = -5 \quad \frac{\partial\phi}{\partial z} = 0.$

$$\therefore \nabla\phi = 12.5i - 5j + 0.k.$$

Given that $\frac{x-1}{2} = \frac{y-3}{-2} = \frac{z-0}{1}.$

The direction of given line is $2i - 2j + 1k = \bar{a}$.

$\therefore$ Directional Derivative $= \nabla\phi \cdot \frac{\bar{a}}{|\bar{a}|}$

$$= (12.5i - 5j + 0k) \cdot \frac{2i - 2j + 1k}{\sqrt{4+4+1}}$$

$$= \frac{25+10}{3} = \frac{35}{3}.$$

Find the values of constants a, b, c so that the directional derivative of $f = axy^2 + byz + cz^2x^3$ at $(1, 2, -1)$ has max. magnitude 64 in the direction parallel to the z-axis.

Sol:- Given that $f = axy^2 + byz + cz^2x^3$.

$$\nabla f = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}.$$

$$\nabla f = i(ay^2 + 3cx^2z^2) + j(2axy + bz) + k(by + 2czx^3)$$

At the point $(1, 2, -1)$

$$\nabla f = i(4a+3c) + j(4a-b) + k(2b-2c) \quad \text{—— (1)}$$

Given that Max. magnitude $= 64$ i.e $|\nabla f| = 64$.

$$\Rightarrow \sqrt{(4a+3c)^2 + (4a-b)^2 + (2b-2c)^2} = 64$$

$$(4a+3c)^2 + (4a-b)^2 + (2b-2c)^2 = 64^2 \quad \text{—— (2)}$$

We find directional derivative of f in the direction of parallel to z-axis. i.e perpendicular to x-axis and y-axis.

Along x-axis unit vector is $\bar{a} = i$

$$\therefore \nabla f.\bar{a} = 0 \Rightarrow \nabla f.\bar{a} = \nabla f.i = 4a+3c = 0 \quad \text{—— (3)}$$

Along y-axis unit vector is $\bar{a} = j$

$$\nabla f.\bar{a} = 0 \Rightarrow \nabla f.\bar{a} = 4a - b = 0 \quad \text{—— (4)}$$

Sub. (3) and (4) in (2), we get

$$b - c = 32 \quad \text{—— (5)}$$

Solving (3), (4) and (5), we get.

$$\therefore a = 6$$
$$b = 24$$
$$c = -8$$

Divergence of a vector :-

Let $\bar{f}$ be any continuously differentiable vector point function. Then $\bar{i}\cdot\frac{\partial \bar{f}}{\partial x}+\bar{j}\cdot\frac{\partial \bar{f}}{\partial y}+\bar{k}\cdot\frac{\partial \bar{f}}{\partial z}$ is called the divergence of $\bar{f}$ and is written as $\operatorname{div}\bar{f}$

i.e $\operatorname{div}\bar{f}=\bar{i}\cdot\frac{\partial \bar{f}}{\partial x}+\bar{j}\cdot\frac{\partial \bar{f}}{\partial y}+\bar{k}\cdot\frac{\partial \bar{f}}{\partial z}=\left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)\cdot\bar{f}$

Hence we can write $\operatorname{div}\bar{f}$ as $\operatorname{div}\bar{f}=\nabla\cdot\bar{f}$

This is a scalar point function.

Note :-

(i) If the vector function $\bar{f}=f_1 i+f_2 j+f_3 k$ then $\operatorname{div}\bar{f}=\frac{\partial f_1}{\partial x}+\frac{\partial f_2}{\partial y}+\frac{\partial f_3}{\partial z}$

(ii) If $\bar{f}$ is a constant vector then $\frac{\partial f_1}{\partial x},\frac{\partial f_2}{\partial y},\frac{\partial f_3}{\partial z}$ are zeros.

$\therefore$ $\operatorname{div}\bar{f}=0$ for a constant vector $\bar{f}$

(iii) $\operatorname{div}(\bar{f}\pm\bar{g})=\operatorname{div}\bar{f}\pm\operatorname{div}\bar{g}$

(iv) $\nabla\cdot\bar{f}\neq\bar{f}\cdot\nabla$

Solenoidal vector :-

A vector point function $\bar{f}$ is said to be solenoidal if $\operatorname{div}\bar{f}=0$.

This equation is also called the equation of continuity or conservation of mass.

1) If $\bar{f}=(x+3y)i+(y-2z)j+(x+pz)k$ is solenoidal, find p.

Sol:- Let $\bar{f}=(x+3y)i+(y-2z)j+(x+pz)k$.

Let $\bar{f}=f_1 i+f_2 j+f_3 k$.

$f_1=x+3y \quad f_2=y-2z \quad f_3=x+pz$

$\frac{\partial f_1}{\partial x}=1 \quad \frac{\partial f_2}{\partial y}=1 \quad \frac{\partial f_3}{\partial z}=p$

$\operatorname{div}\bar{f}=\frac{\partial f_1}{\partial x}+\frac{\partial f_2}{\partial y}+\frac{\partial f_3}{\partial z}=1+1+p=2+p$

Since $\bar{f}$ is solenoidal we have $\operatorname{div}\bar{f}=0 \Rightarrow 2+p=0$

$p=-2$

2) Find $\operatorname{div} \bar{f}$ when $\bar{f} = \operatorname{grad}(x^3+y^3+z^3-3xyz)$

sol:- Let $\phi = x^3+y^3+z^3-3xyz$.

$$\operatorname{grad}\phi = i\frac{\partial \phi}{\partial x} + j\frac{\partial \phi}{\partial y} + k\frac{\partial \phi}{\partial z}.$$

$$\frac{\partial \phi}{\partial x} = 3x^2-3yz \qquad \frac{\partial \phi}{\partial y} = 3y^2-3xz \qquad \frac{\partial \phi}{\partial z} = 3z^2-3xy.$$

$$\bar{f} = \operatorname{grad}\phi = (3x^2-3yz)i + (3y^2-3xz)j + (3z^2-3xy)k = f_1 i + f_2 j + f_3 k.$$

$$\operatorname{div}\bar{f} = i\frac{\partial f_1}{\partial x} + j\frac{\partial f_2}{\partial y} + k\frac{\partial f_3}{\partial z}.$$

$$= \frac{\partial}{\partial x}(3x^2-3yz) + \frac{\partial}{\partial y}(3y^2-3xz) + \frac{\partial}{\partial z}(3z^2-3xy)$$

$$= 6x+6y+6z.$$

(3) If $f = (x^2+y^2+z^2)^{-n}$ then find $\operatorname{div}\operatorname{grad} f$ and determine n if $\operatorname{div}\operatorname{grad} f = 0$.

sol:- Given that $f = (x^2+y^2+z^2)^{-n}$.

Wkt $\operatorname{grad} f = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}$.

$$\frac{\partial f}{\partial x} = -2xn(x^2+y^2+z^2)^{-n-1} \quad \frac{\partial f}{\partial y} = -2yn(x^2+y^2+z^2)^{-n-1} \quad \frac{\partial f}{\partial z} = -2zn(x^2+y^2+z^2)^{-n-1}$$

$$\operatorname{grad} f = (x^2+y^2+z^2)^{-n-1}(-2xni - 2ynj - 2znk)$$

$$\operatorname{grad} f = -2n(x^2+y^2+z^2)^{-n-1}(xi+jy+kz)$$

Wkt $\operatorname{div} f = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$.

Let $f_1 = -2nx(x^2+y^2+z^2)^{-n-1}$ $\quad f_2 = -2ny(x^2+y^2+z^2)^{-n-1}$ $\quad f_3 = -2nz(x^2+y^2+z^2)^{-n-1}$

$$\frac{\partial f_1}{\partial x} = -2n(x^2+y^2+z^2)^{-n-1} + 2n(n+1).2x^2(x^2+y^2+z^2)^{-n-2}$$

$$\frac{\partial f_2}{\partial y} = -2n(x^2+y^2+z^2)^{-n-1} + 4n(n+1)y^2(x^2+y^2+z^2)^{-n-2}$$

$$\frac{\partial f_3}{\partial z} = -2n(x^2+y^2+z^2)^{-n-1} + 4n(n+1)z^2(x^2+y^2+z^2)^{-n-2}$$

$$\operatorname{div}\operatorname{grad} f = -2n(x^2+y^2+z^2)^{-n-1}\left[3 - 2(n+1)(x^2+y^2+z^2)(x^2+y^2+z^2)^{-1}\right]$$

$$\operatorname{div}\operatorname{grad} f = -2n(x^2+y^2+z^2)^{-n-1}(-2n+1).$$

$$= 2n(2n+1)$$

$\operatorname{div}\operatorname{grad} f = 0$ when $n=0$, $n=\frac{1}{2}$

Find $\operatorname{div}\bar{f}$ where $\bar{f} = r^n \bar{r}$ Find n if it is solenoidal. [OR]

Prove that $r^n \bar{r}$ is solenoidal if $n = -3$

Sol:- Given that $\bar{f} = r^n \bar{r}$ where $\bar{r} = xi + yj + zk$

$$r = |\bar{r}| = \sqrt{x^2+y^2+z^2}$$

$$r^2 = x^2+y^2+z^2$$

$$r = (x^2+y^2+z^2)^{1/2}$$

$$r^n = (x^2+y^2+z^2)^{n/2}$$

$$\bar{f} = (x^2+y^2+z^2)^{n/2}(xi+yj+zk)$$

$$\bar{f} = x(x^2+y^2+z^2)^{n/2}.i + y(x^2+y^2+z^2)^{n/2} j + z(x^2+y^2+z^2)^{n/2} k.$$

Let $\bar{f} = f_1 i + f_2 j + f_3 k$.

Here $f_1 = x(x^2+y^2+z^2)^{n/2}$ $\quad f_2 = y(x^2+y^2+z^2)^{n/2}$ $\quad f_3 = z(x^2+y^2+z^2)^{n/2}$

We have $\operatorname{div}\bar{f} = \dfrac{\partial f_1}{\partial x} + \dfrac{\partial f_2}{\partial y} + \dfrac{\partial f_3}{\partial z}$

$$\frac{\partial f_1}{\partial x} = \frac{\partial}{\partial x}\left[x(x^2+y^2+z^2)^{n/2}\right]$$

$$= 1.(x^2+y^2+z^2)^{n/2} + x.\frac{n}{2}(x^2+y^2+z^2)^{\frac{n}{2}-1}.(2x)$$

$$= (x^2+y^2+z^2)^{n/2} + n x^2 (x^2+y^2+z^2)^{\frac{n-2}{2}}$$

$$= (r^2)^{n/2} + n x^2 (r^2)^{\frac{n-2}{2}} \qquad [\because x^2+y^2+z^2 = r^2]$$

$$\frac{\partial f_1}{\partial x} = r^n + n.x^2 r^{n-2}$$

Similarly $\dfrac{\partial f_2}{\partial y} = r^n + n y^2 r^{n-2}$ $\quad \dfrac{\partial f_3}{\partial z} = r^n + n z^2 r^{n-2}$

$$\operatorname{div}\bar{f} = (r^n + n x^2 r^{n-2}) + (r^n + n y^2 r^{n-2}) + (r^n + n z^2 r^{n-2})$$

$$= 3r^n + n r^{n-2}(x^2+y^2+z^2)$$

$$= 3r^n + n r^{n-2}.r^2$$

$\operatorname{div}\bar{f} = 3r^n + nr^n$

$\operatorname{div}\bar{f} = (3+n)r^n$

Let $\bar{f} = r^n \bar{r}$ be solenoidal Then $\operatorname{div}\bar{f} = 0$.

$\therefore (n+3)r^n = 0$

$n = -3$.

Curl of a vector :-

Let $\bar{f}$ be any continuously differentiable vector point function. Then the vector function defined by $i\times\frac{\partial \bar{f}}{\partial x}+j\times\frac{\partial \bar{f}}{\partial y}+k\times\frac{\partial \bar{f}}{\partial z}$ is called curl of $\bar{f}$ and is denoted by curl $\bar{f}$ or $(\nabla\times\bar{f})$

$$\therefore \text{curl}\,\bar{f} = i\times\frac{\partial \bar{f}}{\partial x}+j\times\frac{\partial \bar{f}}{\partial y}+k\times\frac{\partial \bar{f}}{\partial z}.$$

[OR]

If $\bar{f}$ is a differentiable vector function then curl $\bar{f}$ is defined as.

curl $\bar{f} = \nabla\times\bar{f}$.

If $\bar{f} = f_1 i + f_2 j + f_3 k$ then $\text{curl}\,\bar{f} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f_1 & f_2 & f_3 \end{vmatrix}$

$$\text{curl}\,\bar{f} = i\left(\frac{\partial f_3}{\partial y}-\frac{\partial f_2}{\partial z}\right)-j\left(\frac{\partial f_3}{\partial x}-\frac{\partial f_1}{\partial z}\right)+k\left(\frac{\partial f_2}{\partial x}-\frac{\partial f_1}{\partial y}\right).$$

Note:-

(i) The curl of a vector is also a vector.

(ii) If $\bar{f}$ is a constant vector then curl $\bar{f}=\bar{0}$.

(iii) $\text{Curl}(\bar{f}\pm\bar{g}) = \text{curl}\,\bar{f}\pm\text{curl}\,\bar{g}$.

Irrotational Motion, Irrotational Vector :-

Any motion in which curl of the velocity vector is a null vector i.e curl $\bar{v}=\bar{0}$ is said to be irrotational.

→ A vector $\bar{f}$ is said to be irrotational if curl $\bar{f}=\bar{0}$

→ If $\bar{f}$ is irrotational, there will always exists a scalar function $\phi(x,y,z)$ such that $\bar{f}=\text{grad}\,\phi$. This ϕ is called scalar potential of $\bar{f}$.

It is easy to prove that, if $\bar{f}=\text{grad}\,\phi$, then curl $\bar{f}=\bar{0}$.

Hence $\nabla\times\bar{f}=\bar{0} \Longleftrightarrow$ there exists a scalar function ϕ such that $\bar{f}=\nabla\phi$.

This idea is useful when we study the "work done by a force".

1) If $\bar{F} = xy\,i + yz\,j + zx\,k$ find curl $\bar{F}$ at $(1, 2, -3)$.

Sol:- Given that $\bar{F} = xy\,i + yz\,j + zx\,k$. i.e $\bar{F} = f_1 i + f_2 j + f_3 k$

$$\text{curl } \bar{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f_1 & f_2 & f_3 \end{vmatrix}$$

Here $f_1 = xy$

$f_2 = yz$

$f_3 = zx$

$$\text{curl } \bar{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy & yz & zx \end{vmatrix}$$

$$= i\left[\frac{\partial}{\partial y}(zx) - \frac{\partial}{\partial z}(yz)\right] - j\left[\frac{\partial}{\partial x}(zx) - \frac{\partial}{\partial z}(xy)\right] + k\left[\frac{\partial}{\partial x}(yz) - \frac{\partial}{\partial y}(xy)\right]$$

$$= i(0-y) - j(z-0) + k(0-x)$$

$$\text{curl } \bar{F} = -yi - zj - xk$$

At the point $(1, 2, -3)$ curl $\bar{F} = -2i + 3j - k$.

2) Find curl $\bar{F}$ where $\bar{F} = \text{grad}(x^3+y^3+z^3-3xyz)$ (or) $\bar{F} = \nabla(x^3+y^3+z^3-3xyz)$

Sol:- Let $\phi = x^3+y^3+z^3-3xyz$

$$\bar{F} = \text{grad}\,\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$

$$\bar{F} = i(3x^2-3yz) + j(3y^2-3xz) + \bar{k}(3z^2-3xy)$$

$$\text{curl } \bar{F} = \text{curl grad }\phi = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 3x^2-3yz & 3y^2-3xz & 3z^2-3xy \end{vmatrix}$$

$$= i(-3x+3x) - j(-3y+3y) + k(-3z+3z)$$

$$= \bar{0}$$

$\therefore$ curl grad $\phi = \bar{0}$

3) Find constants a, b, c so that the vector $\bar{F} = (x+2y+az)i + (bx-3y-z)j + (4x+cy+2z)k$ is irrotational. Also find ϕ such that $\bar{F} = \nabla\phi$.

Sol: Given that $\bar{F} = (x+2y+az)i + (bx-3y-z)j + (4x+cy+2z)k$.

If $\bar{F} = f_1 i + f_2 j + f_3 k$ then $\text{Curl}\,\bar{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f_1 & f_2 & f_3 \end{vmatrix}$

WKT If $\bar{F}$ is irrotational then $\text{curl}\,\bar{F} = \bar{0}$

Here $f_1 = x+2y+az$ $\quad f_2 = bx-3y-z$ $\quad f_3 = 4x+cy+2z$

$\text{curl}\,\bar{F} = \bar{0}$ i.e $\begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix} = \bar{0}$.

$$i\left[\frac{\partial}{\partial y}(4x+cy+2z) - \frac{\partial}{\partial z}(bx-3y-z)\right] - j\left[\frac{\partial}{\partial x}(4x+cy+2z) - \frac{\partial}{\partial z}(x+2y+az)\right]$$
$$+ k\left[\frac{\partial}{\partial x}(bx-3y-z) - \frac{\partial}{\partial y}(x+2y+az)\right] = 0i+0j+0k$$

$$i(c+1) - j(4-a) + k(b-2) = 0i+0j+0k.$$

$c+1 = 0 \Rightarrow c = -1$.

$a-4 = 0 \Rightarrow a = 4$

$b-2 = 0 \Rightarrow b = 2$.

$\therefore \bar{F} = (x+2y+4z)i + (2x-3y-z)j + (4x-y+2z)k$.

Here $\bar{F}$ is irrotational.

If $\bar{F}$ is irrotational. then there exists a function ϕ such that $\bar{F} = \nabla\phi$.. ϕ is called scalar potential.

We have $\bar{F} = \nabla\phi$

i.e $\bar{F} = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$.

$(x+2y+4z)\,i + (2x-3y-z)\,j + (4x-y+2z)\,k = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$

Comparing bothsides, we have.

$$\frac{\partial\phi}{\partial x} = x+2y+4z \quad \text{——} \ (1)$$

$$\frac{\partial\phi}{\partial y} = 2x-3y-z \quad \text{——} \ (2)$$

$$\frac{\partial\phi}{\partial z} = 4x+y+2z \quad \text{——} \ (3).$$

Integrating (1) w.r.t 'x' partially (treating y and z as constant),

we get $\quad \phi = \frac{x^2}{2} + 2xy + 4xz + c_1 \quad \text{——} \ (4)$

Integrating (2) w.r.t 'y' partially (treating x and z as constant)

we get $\quad \phi = 2xy - 3\frac{y^2}{2} - yz + c_2 \quad \text{——} \ (5)$

Integrating (3) w.r.t 'z' partially (treating x and y as constant)

we get $\quad \phi = 4xz - yz + z^2 + c_3 \quad \text{——} \ (6).$

From (4), (5) and (6),

Hence $\phi = \frac{x^2}{2} - \frac{3y^2}{2} + z^2 + 2xy - yz + 4xz + c.$

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4) Show that the vector $(x^2-yz)i+(y^2-zx)j+(z^2-xy)k$ is irrotational and find its scalar potential.

Sol:- Let $\bar{F} = (x^2-yz)i+(y^2-zx)j+(z^2-xy)k = f_1 i + f_2 j + f_3 k$.

Here $f_1 = x^2-yz \quad f_2 = y^2-zx \quad f_3 = z^2-xy$

$$\text{Curl}\,\bar{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$\text{Curl}\,\bar{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2-yz & y^2-zx & z^2-xy \end{vmatrix}$$

$$= i\left[\frac{\partial}{\partial y}(z^2-xy) - \frac{\partial}{\partial z}(y^2-zx)\right] - j\left[\frac{\partial}{\partial x}(z^2-xy) - \frac{\partial}{\partial z}(x^2-yz)\right] + k\left[\frac{\partial}{\partial x}(y^2-zx) - \frac{\partial}{\partial y}(x^2-yz)\right]$$

$$= i(-x+x) - j(-y+y) + k(-z+z)$$

$\text{Curl}\,\bar{F} = \bar{0}$

$\therefore$ F is irrotational.

If $\bar{F}$ is irrotational then there exists a function ϕ such that $\bar{F} = \nabla\phi$. ϕ is called scalar potential.

We have $\bar{F} = \nabla\phi$ i.e $\bar{F} = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$

$$(x^2-yz)i + (y^2-zx)j + (z^2-xy)k = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$

Comparing components bothsides, we get

$$\frac{\partial\phi}{\partial x} = x^2-yz \quad \text{—①}$$

$$\frac{\partial\phi}{\partial y} = y^2-zx \quad \text{—②}$$

$$\frac{\partial \phi}{\partial z} = z^2 - xy \quad \text{--- (3)}$$

Integrating (1) w.r.t x partially (treating y and z as constant)

we get $\phi = \int (x^2 - yz)\,dx + c_1$

$$\phi = \frac{x^3}{3} - xyz + c_1 \quad \text{--- (4)}$$

Integrating (2) w.r.t y partially (treating x and z as constant)

we get $\phi = \int (y^2 - zx)\,dy + c_2$

$$\phi = \frac{y^3}{3} - xyz + c_2 \quad \text{--- (5)}$$

Integrating (3) w.r.t z partially (treating x and y as constant)

we get $\phi = \int (z^2 - xy)\,dz$

$$\phi = \frac{z^3}{3} - xyz + c_3 \quad \text{--- (6)}$$

From (4) (5) and (6).

$$\phi = \frac{x^3}{3} + \frac{y^3}{3} + \frac{z^3}{3} - xyz + c.$$

Which is the required scalar potential ϕ.

5) If $\bar{F} = (4x+3y+az)i + (bx-y+z)j + (2x+cy+z)k$ is irrotational. Find the constants a, b, c. Also find ϕ such that $\bar{F} = \nabla\phi$.

Sol:- Given that $\bar{F} = (4x+3y+az)i + (bx-y+z)j + (2x+cy+z)k$; $\bar{F} = f_1 i + f_2 j + f_3 k$

Given that $\bar{F}$ is irrotational then curl $\bar{F} = \bar{0}$

i.e
$$\begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 4x+3y+az & bx-y+z & 2x+cy+z \end{vmatrix} = \bar{0}$$

$$i(c-1) - j(2-a) + k(b-3) = 0i + 0j + 0k.$$

Equating the corresponding components, we get

$c-1=0 \Rightarrow c=1$; $a-2=0 \Rightarrow a=2$; $b-3=0 \Rightarrow b=3$

$\therefore a=2, b=3, c=1.$

$\therefore \bar{F} = (4x+3y+2z)i + (3x-y+z)j + (2x+y+z)k$.

If $\bar{F}$ is irrotational then there exists a function ϕ such that $\bar{F} = \nabla\phi$.

ϕ is called scalar potential.

We have $\bar{F} = \nabla\phi$ i.e $\bar{F} = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$

$(4x+3y+2z)i + (3x-y+z)j + (2x+y+z)k = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$

Comparing the corresponding components bothsides, we get

$$\frac{\partial\phi}{\partial x} = 4x+3y+2z \quad \text{—— (1)}$$

$$\frac{\partial\phi}{\partial y} = 3x-y+z \quad \text{—— (2)}$$

$$\frac{\partial\phi}{\partial z} = 2x+y+z \quad \text{—— (3)}$$

Integrating (1) w.r.t 'x' partially (treating y and z as constant)

we get $\phi = \int (4x+3y+2z)\,dx + c_1$

$$\phi = 2x^2 + 3xy + 2xz + c_1 \quad \text{—— (4)}$$

Integrating (2) w.r.t 'y' partially (treating x and z as constant)

we get $\quad \phi = \int (3x - y + z)\, dy + C_2$

$$\phi = 3xy - \frac{y^2}{2} + yz + C_2 \quad \text{--- (5)}$$

Integrating (3) w.r.t 'z' partially treating (x and y as constant)

we get $\quad \phi = \int (2x + y + z)\, dz + C_3$

$$\phi = 2xz + yz + \frac{z^2}{2} + C_3 \quad \text{--- (6)}$$

From (4) (5) and (6)

$$\phi = 2x^2 - \frac{y^2}{2} + \frac{z^2}{2} + 3xy + 2xz + yz + C.$$

Which is the required scalar potential ϕ.

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Vector Integration

Indefinite Integral :-

Integration is the inverse operation of Differentiation.

Let $\bar{F}(t)$ be a differentiable vector function of a scalar variable t and let $\frac{d}{dt}\{\bar{F}(t)\} = \bar{f}(t)$. Then $\int \bar{f}(t)\,dt = \bar{F}(t)$ and $\bar{F}(t)$ is called the primitive of $\bar{f}(t)$. The set of all primitives of $\bar{f}(t)$, that is $\int \bar{f}(t)\,dt = \bar{F}(t)+\bar{c}$ where $\bar{c}$ is any arbitrary constant vector, is called indefinite integral of $\bar{f}(t)$

Hence the indefinite integral of $\bar{f}(t)$ is not unique.

Properties :-

(i) $\int k\bar{f}(t)\,dt = k\int \bar{f}(t)\,dt$, k is a real constant.

(ii) $\int [\bar{f}(t) \pm \bar{g}(t)]\,dt = \int \bar{f}(t)\,dt \pm \int \bar{g}(t)\,dt$

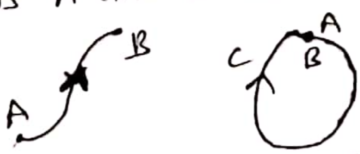
(iii) If $\bar{f}(t) = f_1(t)\bar{i} + f_2(t)\bar{j} + f_3(t)\bar{k}$ then

$$\int \bar{f}(t)\,dt = \bar{i}\int f_1(t)\,dt + \bar{j}\int f_2(t)\,dt + \bar{k}\int f_3(t)\,dt + \bar{c}$$

Definite Integral :- Let $\int \bar{f}(t)\,dt = \bar{F}(t)+\bar{c}$ Then $\int_a^b \bar{f}(t)\,dt = \bar{F}(b) - \bar{F}(a)$.

This is called the Definite Integral of $\bar{f}(t)$ between the limits $t=a$ and $t=b$.

Closed Curve :- Let C be a curve in space. Let A be the initial point and B be the terminal point of the curve C. When the direction along C oriented from A to B is positive then the direction from B to A is called negative direction. If the two points A and B coincide the curve C is called the closed curve.



Smooth Curve :- A curve $\bar{r} = \bar{f}(t)$ is called a smooth curve if $\bar{f}(t)$ is continuously differentiable. A curve C is said to be piecewise smooth if it is the union of finite number of smooth curves.

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Line Integrals :-

Let $\bar{F}(x,y,z)$ be a continuous vector function defined in the entire region of space. Let c be any curve in the region Divide C into n intervals by taking points $A = P_0, P_1, P_2, \ldots P_n = B$.

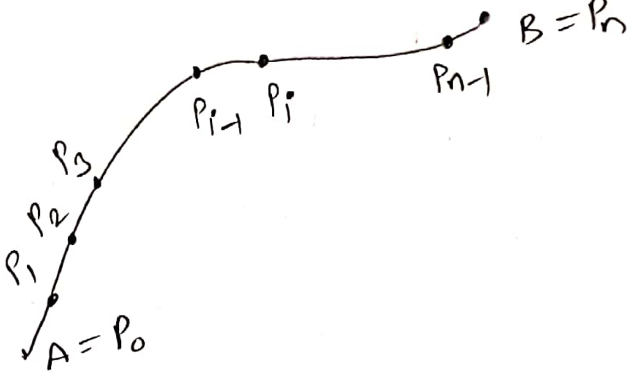
Let B_i be any point in the interval $P_{i-1}P_i$

Let $r_0, r_1, r_2, \ldots r_n$ be the position vectors of the points $P_0, P_1, P_2 \ldots P_n$ respectively. Let us consider the sum. $\Sigma \bar{F}(B_i)\, \delta r_i$

The limit of this sum as $n \to \infty$ and $|\delta r_i| \to 0$ is defined as the line integral of $\bar{F}$ along the curve C and is denoted symbolically by.

$$\int_c \bar{F} \cdot d\bar{r} \quad (\text{or}) \quad \int_c \bar{F} \cdot \frac{d\bar{r}}{dt}\, dt$$ which is a scalar

If C is a closed curve, the integral is written as $\oint \bar{F} \cdot d\bar{r}$



Cartesian form of line integral :-

If $\bar{F} = f_1 i + f_2 j + f_3 k$, $d\bar{r} = dx\, i + dy\, j + dz\, k$

$$\int_c \bar{F} \cdot d\bar{r} = \int_c f_1\, dx + f_2\, dy + f_3\, dz.$$

Note :- $\int_c \phi\, d\bar{r}$ and $\int \bar{F} \times d\bar{r}$ are also examples of line integrals.

In general any integral which is to be evaluated along a curve is called a line integral.

Circulation :- If $\bar{v}$ represents the velocity of a fluid particle and c is a closed curve, then the integral $\oint_c \bar{v} \cdot d\bar{r}$ is called the circulation of $\bar{v}$ around the curve c.

If $\int_c \bar{v} \cdot d\bar{r} = 0$ then the field $\bar{v}$ is called conservative. i.e no work is done. and the energy is conserved.

If the circulation of $\bar{v}$ around every closed curve in a region D vanishes then $\bar{v}$ is said to be irrotational in D. (i.e curl $\bar{v} = \bar{0}$)

Physical applications :-

If $\bar{F}$ represents the force vector acting on a particle moving along an arc AB, then the work done during a small displacement $\delta\bar{r}$ is $\bar{F} \cdot \delta\bar{r}$. Hence the total work done by $\bar{F}$ during displacement from A to B is given by the line integral $\int_A^B \bar{F} \cdot d\bar{r}$

If the force $\bar{F}$ is conservative i.e $\bar{F} = \nabla\phi$ then the work done is independent of the path and vice versa. In this case curl $\bar{F}$ = curl(grad ϕ) = $\bar{0}$ and ϕ is called scalar potential.

Note :- (i) $\bar{F}$ is conservative force field if $\nabla \times \bar{F} = \bar{0}$

(ii) A conservative force field is also irrotational i.e $\nabla \times \bar{F} = \bar{0}$

TYPE-1

1) Using the line integral calculate the work done by the force $\bar{F} = (3x^2+6y)\bar{i} - 14yz\bar{j} + 20xz^2\bar{k}$ along the lines from $(0,0,0)$ to $(1,0,0)$ then to $(1,1,0)$ and then to $(1,1,1)$.

Sol:- Given that $\bar{F} = (3x^2+6y)i - 14yzj + 20xz^2\bar{k}$

Given that the points $O(0,0,0)$ $A(1,0,0)$ $B(1,1,0)$ $C(1,1,1)$

We know that work done by the force $= \int_C \bar{F}\cdot d\bar{r}$

$$\text{Let } \bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$$

$$d\bar{r} = dx\,i + dy\,j + dz\,k$$

$$\bar{F}\cdot d\bar{r} = \left[(3x^2+6y)\bar{i} - 14yz\bar{j} + 20xz^2\bar{k}\right]\cdot(dx\,i + dy\,j + dz\,k)$$

$$\bar{F}\cdot d\bar{r} = (3x^2+6y)dx - 14yz\,dy + 20xz^2\,dz \quad \text{--- (1)}$$

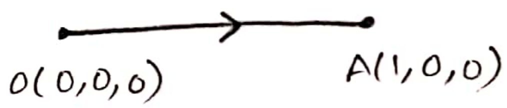
$$\text{Work } W = \int_C \bar{F}\cdot d\bar{r} = \int_{OA}\bar{F}\cdot d\bar{r} + \int_{AB}\bar{F}\cdot d\bar{r} + \int_{BC}\bar{F}\cdot d\bar{r} \quad \text{--- (2)}$$

__Case (i) To evaluate $\int_{OA}\bar{F}\cdot d\bar{r}$ (or) Along the line OA :-__

We have $O(0,0,0)$ $A(1,0,0)$.

Here $y=0$, $z=0$.

$dy=0$ $dz=0$.



x varies from 0 to 1.

$\therefore$ x limits are $x=0$, $x=1$

From (1), $\bar{F}\cdot d\bar{r} = 3x^2\,dx$.

$$\int_{OA}\bar{F}\cdot d\bar{r} = \int_{OA} 3x^2\,dx$$

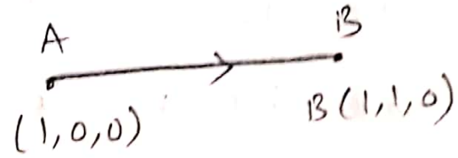
$$= \int_{x=0}^{x=1} 3x^2\,dx = \left[3\frac{x^3}{3}\right]_{x=0}^{x=1}$$

$$= (1-0)$$

$$\int_{OA}\bar{F}\cdot d\bar{r} = 1$$

Case (ii) To evaluate $\int_{AB} \bar{F}\cdot d\bar{r}$ (or) Along the line AB :-

We have $A(1,0,0)$ $B(1,1,0)$.



Here $x=1$, $z=0$

$dx=0$ $dz=0$

y varies from 0 to 1.

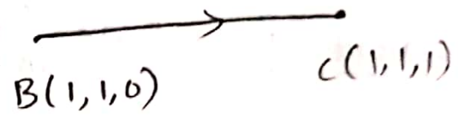
$\therefore$ y limits $y=0$, $y=1$.

From ①, $\bar{F}\cdot d\bar{r} = 0$

$$\int_{AB} \bar{F}\cdot d\bar{r} = 0. \quad \text{——} ④.$$

Case (iii) To evaluate $\int_{BC} \bar{F}\cdot d\bar{r}$ (or) Along the line BC :-

We have $B(1,1,0)$ $C(1,1,1)$



$x=1$ $y=1$

$dx=0$ $dy=0$

z varies from 0 to 1.

$\therefore$ z limits $z=0$, $z=1$.

From ①, $\bar{F}\cdot d\bar{r} = 20z^2\,dz$.

$$\int_{BC} \bar{F}\cdot d\bar{r} = \int_{BC} 20z^2\,dz$$

$$= \int_{z=0}^{z=1} 20z^2\,dz$$

$$= \left[20\frac{z^3}{3}\right]_{z=0}^{z=1}$$

$$= \frac{20}{3} \quad \text{——} ⑤$$

Sub. ③, ④ and ⑤ in ②, we get

$$\int_C \bar{F}\cdot d\bar{r} = 1+0+\frac{20}{3} = \frac{23}{3}$$

2 If $F = (x^2-27)i - 6yz\,j + 8xz^2\,k$ evaluate $\int_c F\cdot d\bar{r}$ from the point $(0,0,0)$ to the point $(1,1,1)$ along the straight line from $(0,0,0)$ to $(1,0,0)$, $(1,0,0)$ to $(1,1,0)$ and $(1,1,0)$ to $(1,1,1)$.

Ans:- $-\frac{72}{3}$

Sol:- Given that $\bar{F} = (x^2-27)i - 6yz\,j + 8xz^2\,k$.

Given that the points $O(0,0,0)$ $A(1,0,0)$ $B(1,1,0)$ $C(1,1,1)$.

We have to find $\int_c \bar{F}\cdot d\bar{r}$.

$$\text{Let } \bar{r} = xi + yj + zk.$$

$$d\bar{r} = dx\,i + dy\,j + dz\,k.$$

$$\bar{F}\cdot d\bar{r} = \left[(x^2-27)i - 6yz\,j + 8xz^2\,k\right]\cdot\left[dx\,i + dy\,j + dz\,k\right]$$

$$\bar{F}\cdot d\bar{r} = (x^2-27)\,dx - 6yz\,dy + 8xz^2\,dz \quad \text{——} \ (1)$$

$$\int_c \bar{F}\cdot d\bar{r} = \int_{OA} \bar{F}\cdot d\bar{r} + \int_{AB} \bar{F}\cdot d\bar{r} + \int_{BC} \bar{F}\cdot d\bar{r} \quad \text{——} \ (2)$$

Case (i) To evaluate $\int_{OA} \bar{F}\cdot d\bar{r}$ (or) Along the line OA :-

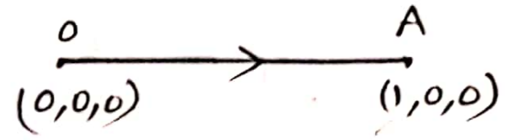
We have $O(0,0,0)$ $A(1,0,0)$

Here $y=0$ $z=0$

$dy=0$ $dz=0$

x varies from 0 to 1.

$\therefore$ x limits $x=0$, $x=1$.



From (1), $\bar{F}\cdot d\bar{r} = (x^2-27)\,dx$.

$$\int_{OA} \bar{F}\cdot d\bar{r} = \int_{OA} (x^2-27)\,dx$$

$$= \int_{x=0}^{x=1} (x^2-27)\,dx = \left[\frac{x^3}{3} - 27x\right]_{x=0}^{x=1}$$

$$= \left(\frac{1}{3} - 27\right) - 0$$

$$\int_{OA} \bar{F}\cdot d\bar{r} = -\frac{80}{3} \quad \text{——} \ (3)$$

Case(ii) To evaluate $\int_{AB} \bar{F}\cdot d\bar{r}$ (or) Along the line AB :-

We have $A(1,0,0)$ $B(1,1,0)$

Here $x=1$, $z=0$

$dx=0$ $dz=0$

y varies from 0 to 1

$\therefore$ y limits $y=0$, $y=1$

From (1) $\bar{F}\cdot d\bar{r} = 0$

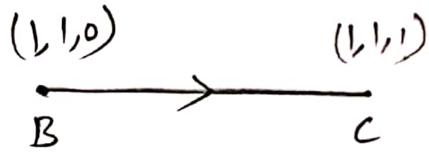
$$\int_{AB} \bar{F}\cdot d\bar{r} = 0 \quad \text{--- (4)}$$

Case(iii) To evaluate $\int_{BC} \bar{F}\cdot d\bar{r}$ (or) Along the line BC :-

We have $B(1,1,0)$ $C(1,1,1)$

Here $x=1$, $y=1$

$dx=0$ $dy=0$



z varies from 0 to 1

$\therefore$ z limits $z=0$, $z=1$

From (1) $\bar{F}\cdot d\bar{r} = 8z^2\,dz$

$$\int_{BC} \bar{F}\cdot d\bar{r} = \int_{BC} 8z^2\,dz$$

$$= \int_{z=0}^{z=1} 8z^2\,dz$$

$$= \left[\frac{8z^3}{3}\right]_{z=0}^{z=1}$$

$$= \frac{8}{3} \quad \text{--- (5)}$$

sub. (3) (4) and (5) in (2), we get

$$\int_C \bar{F}\cdot d\bar{r} = -\frac{80}{3} + \frac{8}{3}$$

$$\int_C \bar{F}\cdot d\bar{r} = \frac{-72}{3}$$

TYPE - 2

→ Find the work done by the force $\bar{F} = (3x^2 - 6yz)i + (2y + 3xz)j + (1 - 4xyz^2)k$ in moving particle from the point $(0,0,0)$ to the point $(1,1,1)$ along the curve C : $x = t$ $y = t^2$ $z = t^3$.

Sol: Given that $\bar{F} = (3x^2 - 6yz)\bar{i} + (2y + 3xz)\bar{j} + (1 - 4xyz^2)\bar{k}$

WKT Work done by the force $= \int_C \bar{F} \cdot d\bar{r}$

Let $\bar{r} = xi + yj + zk$.

Given that the curve C is $x = t$ $y = t^2$ $z = t^3$

$dx = dt$ $dy = 2t\,dt$ $dz = 3t^2\,dt$

$d\bar{r} = dx\,i + dy\,j + dz\,k$.

$d\bar{r} = dt\,i + 2t\,dt\,j + 3t^2\,dt\,k$.

$\bar{F} = (3t^2 - 6t^5)i + (2t^2 + 3t^4)j + (1 - 4t^9)k$.

$\bar{F} \cdot d\bar{r} = [(3t^2 - 6t^5)i + (2t^2 + 3t^4)j + (1 - 4t^9)k] \cdot [dt\,i + 2t\,dt\,j + 3t^2\,dt\,k]$

$= (3t^2 - 6t^5)dt + (2t^2 + 3t^4)\,2t\,dt + (1 - 4t^9)\,3t^2\,dt$.

$\bar{F} \cdot d\bar{r} = (-12t^{11} + 4t^3 + 6t^2)\,dt$

The end points are $O(0,0,0)$ $A(1,1,1)$

We have $x = t$ $y = t^2$ $z = t^3$

At $x = 0$, $t = 0$ At $x = 1$, $t = 1$.

At $y = 0$, $t = 0$ At $y = 1$, $t = 1$

At $z = 0$, $t = 0$ At $z = 1$, $t = 1$

∴ The corresponding values of t are $t = 0$ and $t = 1$.

∴ t limits are $t = 0$, $t = 1$

∴ Work $= \int_C \bar{F} \cdot d\bar{r}$

$= \int_C (-12t^{11} + 4t^3 + 6t^2)\,dt = \int_{t=0}^{t=1} (-12t^{11} + 4t^3 + 6t^2)\,dt$

$= \left[-t^{12} + t^4 + 2t^3\right]_{t=0}^{t=1}$

Work. $= 2$

2\. If $\bar{F} = xy\bar{i} - z\bar{j} + x^2\bar{k}$ and c is the curve $x = t^2$, $y = 2t$, $z = t^3$ from $t=0$ to $t=1$. Evaluate $\int_c \bar{F}\cdot d\bar{r}$. Ans:- $\frac{51}{70}$.

Sol:- Given that $F = xy\, i - z\, j + x^2 k$.

Given that the curve $x = t^2$ $y = 2t$ $z = t^3$.

We have to find $\int_c \bar{F}\cdot d\bar{r}$

$$r = xi + yj + zk.$$

$$d\bar{r} = dx\, i + dy\, j + dz\, k$$

$$dx = 2t\,dt \quad dy = 2dt \quad dz = 3t^2\,dt$$

$$d\bar{r} = (2t\, i + 2j + 3t^2 k)\,dt$$

$$\bar{F} = 2t^3\, i - t^3\, j + t^4 k.$$

$$\bar{F}\cdot d\bar{r} = (2t^3 i - t^3 j + t^4 k)\cdot(2t\,dt\, i + 2dt\, j + 3t^2 dt\, k)$$

$$\bar{F}\cdot d\bar{r} = (4t^4 - 2t^3 + 3t^6)\,dt$$

Given that t varies from 0 to 1.

$$\int_c \bar{F}\cdot d\bar{r} = \int_c (4t^4 - 2t^3 + 3t^6)\,dt$$

$$= \int_{t=0}^{t=1} (4t^4 - 2t^3 + 3t^6)\,dt$$

$$= \left[\frac{4t^5}{5} - \frac{2t^4}{4} + \frac{3t^7}{7}\right]_{t=0}^{t=1}$$

$$= \left(\frac{4}{5} - \frac{2}{4} + \frac{3}{7}\right) - 0$$

$$= \frac{112 - 70 + 60}{140}$$

$$\int_c \bar{F}\cdot d\bar{r} = \frac{51}{70}$$

Note:- Let two points be (x_1, y_1, z_1) (x_2, y_2, z_2). Then the equation of the line $\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$ (Symmetric form)

TYPE-3

1) Find the work done in moving a particle in the force field $\bar{F} = 3x^2 i + j + z\bar{k}$ along the straight line from $(0,0,0)$ to $(2,1,3)$.

sol:- Given that $\bar{F} = 3x^2 i + j + zk$

W.k.t the Work done by the force $= \int_c \bar{F} \cdot d\bar{r}$

Let $\bar{r} = xi + yj + zk$.

$d\bar{r} = dx\, i + dy\, j + dz\, k$.

Given that $O(0,0,0)$ $A(2,1,3)$

Equation of OA is $\frac{x-0}{2-0} = \frac{y-0}{1-0} = \frac{z-0}{3-0}$.

$\frac{x}{2} = \frac{y}{1} = \frac{z}{3} = t$ (say)

$x = 2t \quad y = t \quad z = 3t$

$dx = 2dt \quad dy = dt \quad dz = 3dt$

$d\bar{r} = 2dt\, i + dt\, j + 3\, dt\, k$.

The points $(0,0,0)$ and $(2,1,3)$ Correspond to $t=0$ and $t=1$ respectively

$\therefore$ t limits are $t=0$, $t=1$.

We have
$O(0,0,0)$ $A(2,1,3)$
when $x=0, y=0, z=0$.
$\Rightarrow t=0$.
$\therefore x = 2t, y = t, z = 3t$
when $x=2, y=1, z=3$
$\Rightarrow t=1$.

$\bar{F} \cdot d\bar{r} = (3x^2 i + j + zk) \cdot (2dt\, i + dt\, j + 3dt\, k)$

$= 3x^2(2dt) + dt + z(3dt)$

$= 3(2t)^2 2dt + dt + (3t)(3dt)$

$\bar{F} \cdot d\bar{r} = 24t^2\, dt + dt + 9t\, dt.$

$\int_c \bar{F} \cdot d\bar{r} = \int_c 24t^2 dt + 9t\, dt + dt$

$= \int_{t=0}^{t=1} (24t^2\, dt + 9t\, dt + dt) = \left[24 \cdot \frac{t^3}{3} + 9\frac{t^2}{2} + t\right]_{t=0}^{t=1}$

$= 8 + \frac{9}{2} + 1 = \frac{27}{2}$

→ Find the work done in moving a particle in the force field $\bar{F} = 3x^2 i + (2xz-y) j + zk$ along the straight line from $(0,0,0)$ to $(2,1,3)$

Ans:- 16.

Sol:- Given that $\bar{F} = 3x^2 i + (2xz-y) j + zk$.

Wkt Work done by the force $= \int_c \bar{F}.d\bar{r}$

Let $\bar{r} = xi + yj + zk$.

$d\bar{r} = dx\, i + dy\, j + dz\, k$

Given that $O(0,0,0)$ $A(2,1,3)$

Equation of OA is. $\frac{x-0}{2-0} = \frac{y-0}{1-0} = \frac{z-0}{3-0}$.

$\frac{x}{2} = \frac{y}{1} = \frac{z}{3} = t$ (say)

$x = 2t \quad y = t \quad z = 3t$

$dx = 2dt \quad dy = dt \quad dz = 3dt$

$d\bar{r} = (2i + j + 3k) dt$

$\bar{F} = 12t^2 i + (12t^2 - t) j + 3t k$.

$\bar{F}.d\bar{r} = [12t^2 i + (12t^2 - t) j + 3t k] . (2i + j + 3k) dt$

$\bar{F}.d\bar{r} = 24t^2 + 12t^2 - t + 9t = (36t^2 + 8t) dt$.

The points $O(0,0,0)$ $A(2,1,3)$ correspond to $t=0$ and $t=1$ respectively.

∴ t limits are $t=0, t=1$.

We have $x = 2t, y = t, z = 3t$

$O(0,0,0)$ $A(2,1,3)$

when $x=0, y=0, z=0$.

$\Rightarrow t = 0$.

when $x=2, y=1, z=3$.

$t=1$.

$\int_c \bar{F}.d\bar{r} = \int_c (36t^2 + 8t) dt$

$= \int_{t=0}^{t=1} (36t^2 + 8t) dt$

$= \left[36.\frac{t^3}{3} + 8.\frac{t^2}{2}\right]_{t=0}^{t=1}$

$= (12+4) - 0$

$\int_c \bar{F}.d\bar{r} = 16$

→ Prove that force field given by $\bar{F} = 2xyz^3 i + x^2z^3 j + 3x^2yz^2 k$ is Conservative. Find the work done by moving a particle from $(1,-1,2)$ to $(3,2,-1)$ in this force field.

Ans:- -10.

Sol:- If the integral $\int_{P_1}^{P_2} \bar{F}\cdot d\bar{r}$ is independent of the path joining P_1 and P_2 then $\bar{F}$ is called a conservative field.

Alternatively, $\bar{F}$ is called a conservative field if curl $\bar{F} = \bar{0}$ iff there exists a scalar point function ϕ such that $\bar{F} = \nabla\phi$.

Given that $\bar{F} = 2xyz^3 i + x^2z^3 j + 3x^2yz^2 k$ $\quad \bar{F} = F_1 i + F_2 j + F_3 k$

$$\text{curl}\,\bar{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xyz^3 & x^2z^3 & 3x^2yz^2 \end{vmatrix}$$

$$\text{curl}\,\bar{F} = i(3x^2z^2 - 3x^2z^2) - j(6xyz^2 - 6xyz^2) + k(2xz^3 - 2xz^3)$$

$$\text{curl}\,\bar{F} = \bar{0}$$

Hence $\bar{F}$ is conservative.

To find ϕ such that $\bar{F} = \nabla\phi$

$$2xyz^3 i + x^2z^3 j + 3x^2yz^2 k = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$

$$\frac{\partial\phi}{\partial x} = 2xyz^3 \quad \frac{\partial\phi}{\partial y} = x^2z^3 \quad \frac{\partial\phi}{\partial z} = 3x^2yz^2$$

We have $d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$

$$d\phi = 2xyz^3\,dx + x^2z^3\,dy + 3x^2yz^2\,dz$$

$$d\phi = d(x^2yz^3)$$

$$\Rightarrow \phi = x^2yz^3 + c.$$

Work done by the force $= \int_{P_1}^{P_2} \bar{F}\cdot d\bar{r}$

We have $\bar{r} = xi + yj + zk \Longrightarrow d\bar{r} = dx\,i + dy\,j + dz\,k$.

$$\bar{F}\cdot d\bar{r} = 2xyz^3\,dx + x^2z^3\,dy + 3x^2yz^2\,dz = d(x^2yz^3)$$

$$\text{Hence Work} = \int_{P_1}^{P_2} \bar{F}\cdot d\bar{r} = \int_{(1,-1,2)}^{(3,2,-1)} d(x^2yz^3) = \left[x^2yz^3\right]_{(1,-1,2)}^{(3,2,-1)}$$

Work. $= -10$

TYPE-4

1) Find the circulation of $\bar{F} = (2x-y+2z)i + (x+y-z)j + (3x-2y-5z)k$ along the circle $x^2+y^2=4$ in the xy-plane.

Sol:- Given that $F = (2x-y+2z)i + (x+y-z)j + (3x-2y-5z)k$

WKT circulation $= \oint_C \bar{F}.d\bar{r}$ where c is the circle $x^2+y^2=4$

Let $\bar{r} = xi+yj+zk$.

$\bar{r} = xi+yj$ [$\because$ In xy-plane $z=0$]

$d\bar{r} = dx\,i + dy\,j + 0.k$.

$F = (2x-y)i + (x+y)j + (3x-2y)k$ [$\because$ In xy-plane $z=0$]

$\bar{F}.d\bar{r} = [(2x-y)i + (x+y)j + (3x-2y)k].(dx\,i + dy\,j + 0.k)$

$\bar{F}.d\bar{r} = (2x-y)\,dx + (x+y)\,dy$

Given that the circle $x^2+y^2=4$.

$x = 2\cos\theta \quad y = 2\sin\theta$

$dx = -2\sin\theta\,d\theta, \quad dy = 2\cos\theta\,d\theta$

In circle, θ varies from 0 to 2π

$\therefore$ θ limits $\theta=0, \theta=2\pi$

$\bar{F}.d\bar{r} = (4\cos\theta - 2\sin\theta)(-2\sin\theta)d\theta + (2\cos\theta + 2\sin\theta)(2\cos\theta)d\theta$

$\bar{F}.d\bar{r} = [-8\sin\theta\cos\theta + 4\sin^2\theta + 4\cos^2\theta + 4\sin\theta\cos\theta]d\theta$

$= (4 - 4\sin\theta\cos\theta)d\theta$

$\bar{F}.d\bar{r} = (4 - 2\sin2\theta)d\theta$

$\int_C \bar{F}.d\bar{r} = \int_C (4-2\sin2\theta)\,d\theta$

$= \int_{\theta=0}^{\theta=2\pi} (4-2\sin2\theta)d\theta$

$= [4\theta + \cos2\theta]_{\theta=0}^{\theta=2\pi} = 8\pi + \cos4\pi - \cos0.$

$= 8\pi$

2) Find the work done by $\bar{F} = (2x-y-z)i + (x+y-z)j + (3x-2y-5z)k$ along a curve C in the xy-plane given by $x^2+y^2=9,\ z=0$ Ans:- 18π

Sol:- Given that $\bar{F} = (2x-y-z)i + (x+y-z)j + (3x-2y-5z)k$.

Let $\bar{r} = xi + yj + zk$.

$d\bar{r} = dx\,i + dy\,j + dz\,k$.

In xy-plane. $z=0$ $\therefore dz = 0$.

$d\bar{r} = dx\,i + dy\,j$

$\bar{F} = (2x-y)i + (x+y)j + (3x-2y)k$. [$\because$ In xy-plane $z=0$]

$\bar{F}\cdot d\bar{r} = [(2x-y)i + (x+y)j + (3x-2y)k]\cdot[dx\,i + dy\,j + 0\cdot k]$

$\bar{F}\cdot d\bar{r} = (2x-y)dx + (x+y)dy$.

Given that the curve c in the xy-plane is $x^2+y^2=9,\ z=0$.

The parametric equations of circle $x^2+y^2=9$ are $x=3\cos\theta$ $y=3\sin\theta$

$dx = -3\sin\theta\, d\theta$ $dy = 3\cos\theta\, d\theta$

$\therefore$ θ varies from 0 to 2π.

$\therefore$ $\bar{F}\cdot d\bar{r} = (6\cos\theta - 3\sin\theta)(-3\sin\theta)d\theta + (3\cos\theta + 3\sin\theta)(3\cos\theta)d\theta$

$\bar{F}\cdot d\bar{r} = [-18\sin\theta\cos\theta + 9\sin^2\theta + 9\cos^2\theta + 9\sin\theta\cos\theta]d\theta$

$\bar{F}\cdot d\bar{r} = [9 - 9\sin\theta\cos\theta]d\theta$

$$\int_c \bar{F}\cdot d\bar{r} = \int_c (9 - 9\sin\theta\cos\theta)d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi} (9 - 9\sin\theta\cos\theta)d\theta$$

$$= 9[\theta]_{\theta=0}^{\theta=2\pi} - 9\left[\frac{\sin^2\theta}{2}\right]_{\theta=0}^{\theta=2\pi}$$

$$= 9(2\pi - 0) - \frac{9}{2}[\sin^2(2\pi) - \sin(0)]$$

$$\int_c \bar{F}\cdot d\bar{r} = 18\pi$$

→ A vector field is given by $\bar{F} = \sin y\, i + x(1+\cos y)\, j$. Evaluate the line integral over a circular path given by $x^2+y^2=a^2,\ z=0$.

Sol:- Given that $\bar{F} = \sin y\, i + x(1+\cos y)\, j$

We have $\bar{r} = xi + yj$

$d\bar{r} = dx\, i + dy\, j$

$\bar{F}\cdot d\bar{r} = [\sin y\, i + x(1+\cos y) j]\cdot[dx\, i + dy\, j]$

$\bar{F}\cdot d\bar{r} = \sin y\, dx + x(1+\cos y)\, dy$

Given that the circle $x^2+y^2=a^2$

The parametric equations are $x = a\cos\theta \quad y = a\sin\theta$

$dx = -a\sin\theta\, d\theta \quad dy = a\cos\theta\, d\theta$

θ varies from 0 to 2π

$\therefore \theta$ limits $\theta=0,\ \theta=2\pi$.

$$\oint_C \bar{F}\cdot d\bar{r} = \oint_C \sin y\, dx + x(1+\cos y)\, dy.$$

$$= \int_{\theta=0}^{\theta=2\pi} \sin(a\sin\theta)\cdot(-a\sin\theta)\, d\theta + a\cos\theta(1+\cos(a\sin\theta))\, a\cos\theta\, d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi} [-a\sin\theta \sin(a\sin\theta) + a^2\cos^2\theta + a^2\cos^2\theta\cos(a\sin\theta)]\, d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi} a^2\cos^2\theta\, d\theta + \int_{\theta=0}^{\theta=2\pi} [a^2\cos^2\theta\cos(a\sin\theta) - a\sin\theta\sin(a\sin\theta)]\, d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi} a^2\left(\frac{1+\cos 2\theta}{2}\right) + \int_{\theta=0}^{\theta=2\pi} d[a\cos\theta\sin(a\sin\theta)]$$

$$= \frac{a^2}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_{\theta=0}^{\theta=2\pi} + [a\cos\theta\sin(a\sin\theta)]_{\theta=0}^{\theta=2\pi}$$

$$= \frac{a^2}{2}\left[\left(2\pi + \frac{\sin 4\pi}{2}\right) - 0\right] + [a\cos(2\pi)\sin(a\sin(2\pi)) - 0]$$

$$\int_C \bar{F}\cdot d\bar{r} = \pi a^2$$

→ If $\bar{F} = 2yi - zj + xk$, evaluate $\int_C \bar{F}\times d\bar{r}$ along the curve $x = \cos t$, $y = \sin t$, $z = 2\cos t$ from $t=0$ to $t = \pi/2$

Sol: Given that $\bar{F} = 2yi - zj + xk$. i.e $\bar{F} = F_1 i + F_2 j + F_3 k$

$\bar{r} = xi + yj + zk$. Here $F_1 = 2y$

$d\bar{r} = dx\, i + dy\, j + dz\, k$ $F_2 = -z$ $F_3 = x$.

We have to find $\int_C \bar{F}\times d\bar{r}$

$$\bar{F}\times d\bar{r} = \begin{vmatrix} i & j & k \\ F_1 & F_2 & F_2 \\ dx & dy & dz \end{vmatrix}$$

$$= \begin{vmatrix} i & j & k \\ 2y & -z & x \\ dx & dy & dz \end{vmatrix}$$

$$\bar{F}\times d\bar{r} = i(-zdz - xdy) - j(2ydz - xdx) + k(2ydy + zdx) \quad (1)$$

Given that the curve is. $x = \cos t$ $y = \sin t$ $z = 2\cos t$

$dx = -\sin t\, dt$ $dy = \cos t\, dt$ $dz = -2\sin t\, dt$ (2)

sub. (2) in (1), we get

$$\bar{F}\times d\bar{r} = i(4\sin t\cos t - \cos^2 t)dt - j(-4\sin^2 t + \sin t\cos t)dt$$

$$\int_C \bar{F}\times d\bar{r} = \int_C \left[i(4\sin t\cos t - \cos^2 t)\,dt - j(\sin t\cos t - 4\sin^2 t)\,dt\right]$$

$$= i\int_{t=0}^{t=\pi/2}(4\sin t\cos t - \cos^2 t)\,dt + j\int_{t=0}^{t=\pi/2}(4\sin^2 t - \sin t\cos t)\,dt$$

$$= i\int_{t=0}^{t=\pi/2}\left(2\sin 2t - \left(\frac{1+\cos 2t}{2}\right)\right)dt + j\int_{t=0}^{t=\pi/2}\left(2(1-\cos 2t) - \frac{\sin 2t}{2}\right)dt$$

$$= i\left(2 - \frac{\pi}{4}\right) + j\left(\pi - \frac{1}{2}\right).$$

→ Evaluate $\oint_C (yz\,dx + xz\,dy + xy\,dz)$ over arc of a helix $x = a\cos t$, $y = a\sin t$, $z = kt$ as t varies from 0 to 2π Ans:- 0.

Sol:- Given that $I = \oint_C (yz\,dx + xz\,dy + xy\,dz)$

Given that the curve is $x = a\cos t$ $y = a.\sin t$ $z = kt$.

$dx = -a\sin t\,dt$ $dy = a\cos t\,dt$ $dz = k\,dt$.

Given that t varies from 0 to 2π.

$$\oint_C (yz\,dx + xz\,dy + xy\,dz) = \oint_C akt\sin t(-a\sin t)dt + akt\cos t(a\cos t)\,dt + a^2\sin t\cos t(k\,dt)$$

$$= -a^2k\int_{t=0}^{t=2\pi} t\sin^2 t\,dt + a^2k\int_{t=0}^{t=2\pi} t\cos^2 t\,dt + a^2k\int_{t=0}^{t=2\pi} \sin t\cos t\,dt$$

$$= a^2k\int_{t=0}^{t=2\pi} t(\cos^2 t - \sin^2 t)\,dt + a^2k\int_{t=0}^{t=2\pi} \sin t\cos t\,dt$$

$$= a^2k\int_{t=0}^{t=2\pi} t\cos 2t\,dt + a^2k\left[\frac{\sin^2 t}{2}\right]_{t=0}^{t=2\pi}$$

$$= a^2k\left[t\left(\frac{\sin 2t}{2}\right) - 1.\left(\frac{-\cos 2t}{4}\right)\right]_{t=0}^{t=2\pi} + \frac{a^2k}{2}\left[\sin^2(2\pi) - \sin^2(0)\right]$$

$$= a^2k\left[\left\{2\pi\left(\frac{\sin 4\pi}{2}\right) + \frac{\cos 4\pi}{4}\right\} - \left(\frac{\cos 0}{4}\right)\right]$$

$$= 0$$

TYPE - 5

→ If c is the curve $y = 3x^2$ in the xy-plane and $\bar{F} = (x+2y)i - xy\,j$ evaluate $\int_c \bar{F}\cdot d\bar{r}$ from the point $(0,0)$ to $(1,3)$.

Sol:- Given that $\bar{F} = (x+2y)i - xy\,j$

We have to find $\int_c \bar{F}\cdot d\bar{r}$

We have $\bar{r} = xi + yj + zk$

$\bar{r} = xi + yj$ [$\because$ In xy-plane $z=0$]

$d\bar{r} = dx\,i + dy\,j$

$\bar{F}\cdot d\bar{r} = [(x+2y)i - xy\,j]\cdot(dx\,i + dy\,j)$

$\bar{F}\cdot d\bar{r} = (x+2y)\,dx - xy\,dy$

Given that the curve $y = 3x^2$

$dy = 6x\,dx$.

Given that the points $O(0,0)$ $A(1,3)$

$\therefore$ x varies from 0 to 1.

$$\int_c \bar{F}\cdot d\bar{r} = \int_c (x + 6x^2)\,dx - 3x^3 \cdot 6x\,dx.$$

$$= \int_c (x + 6x^2 - 18x^4)\,dx$$

$$= \int_{x=0}^{x=1} (x + 6x^2 - 18x^4)\,dx$$

$$= \left[\frac{x^2}{2} + 2x^3 - \frac{18}{5}x^5\right]_{x=0}^{x=1}.$$

$$= \frac{-11}{10}.$$

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→ If $\bar{F} = (5xy - 6x^2)\,i + (2y - 4x)\,j$ evaluate $\int_c \bar{F}\cdot d\bar{r}$ along the curve c in xy-plane $y = x^3$ from (1,1) to (2,8)

Ans:- 35

Sol:- Given that $\bar{F} = (5xy - 6x^2)\,i + (2y - 4x)\,j$

Given that the curve $y = x^3 \Rightarrow dy = 3x^2\,dx$.

$$\bar{F} = (5x^4 - 6x^2)\,i + (2x^3 - 4x)\,j \quad [\because y = x^3]$$

Let $\bar{r} = xi + yj$.

$$d\bar{r} = dx\,i + dy\,j \quad [\because \text{In xy-plane } z = 0]$$

$$d\bar{r} = dx\,i + 3x^2\,dx\,j$$

$$\bar{F}\cdot d\bar{r} = \left[(5x^4 - 6x^2)\,i + (2x^3 - 4x)\,j\right]\cdot\left[dx\,i + 3x^2\,dx\,j\right]$$

$$\bar{F}\cdot d\bar{r} = (6x^5 + 5x^4 - 12x^3 - 6x^2)\,dx$$

We have A(1,1) B(2,8)

x varies from 1 to 2.

$\therefore$ x limits $x = 1,\ x = 2$.

$$\int_c \bar{F}\cdot d\bar{r} = \int_c (6x^5 + 5x^4 - 12x^3 - 6x^2)\,dx.$$

$$= \int_{x=1}^{x=2} (6x^5 + 5x^4 - 12x^3 - 6x^2)\,dx$$

$$= \left[6\cdot\frac{x^6}{6} + 5\cdot\frac{x^5}{5} - 12\,\frac{x^4}{4} - 6\cdot\frac{x^3}{3}\right]_{x=1}^{x=2}$$

$$= \left[x^6 + x^5 - 3x^4 - 2x^3\right]_{x=1}^{x=2}$$

$$= (2^6 + 2^5 - 3\cdot 2^4 - 2\cdot 2^3) - (1 + 1 - 3 - 2)$$

$$\int_c \bar{F}\cdot d\bar{r} = 35$$

→ Compute the line integral $\int_C (y^2 dx - x^2 dy)$ round the triangle whose vertices are $(1,0)$ $(0,1)$ $(-1,0)$ in the xy-plane.

TYPE – 6

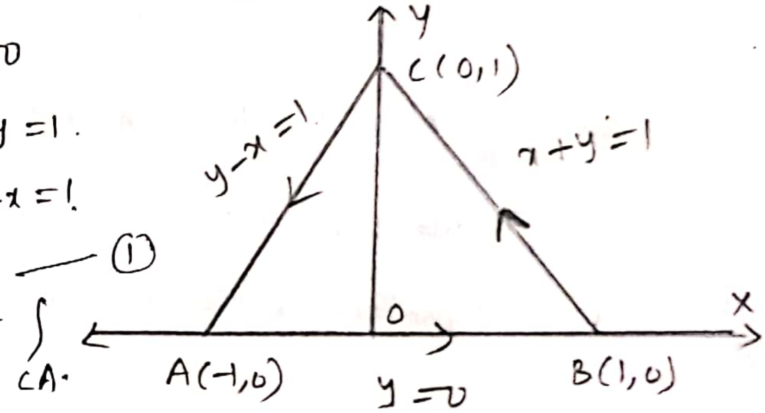
Sol: Let $I = \int_C y^2 dx - x^2 dy$ . .

Let the vertices of the triangle are $A(-1,0)$ $B(1,0)$ $C(0,1)$.

Equation of the line AB is $y = 0$

Equation of the line BC is $x + y = 1$.

Equation of the line CA is $y - x = 1$ —— ①



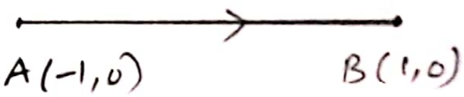
$$\therefore \int_C y^2 dx - x^2 dy = \int_{AB} + \int_{BC} + \int_{CA}.$$

Case (i) :- Along the line AB (or) To evaluate $\int_{AB} y^2 dx - x^2 dy$.

We have $A(-1,0)$ $B(1,0)$

Here $y = 0 \Rightarrow dy = 0$.

x varies from -1 to 1.



$\therefore$ x limits are $x = -1, x = 1$.

$$\int_{AB} y^2 dx - x^2 dy = \int_{AB} 0.dx - x^2.(0)$$

$$\int_{AB} y^2 dx - x^2 dy = 0 \quad \text{—— ②}$$

Case (ii) :- Along the line BC (or) To evaluate $\int_{BC} y^2 dx - x^2 dy$.

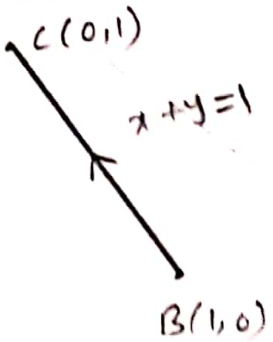
We have $B(1,0)$ $C(0,1)$

Equation of the line BC is $x + y = 1 \Rightarrow y = 1 - x$

$dy = -dx$.

$\therefore$ x varies from 1 to 0

$\therefore$ x limits are $x = 1, x = 0$.



$$\int_{BC} y^2 dx - x^2 dy = \int_{BC} (1-x)^2 dx - x^2(-dx)$$

$$= \int_{x=1}^{x=0} \left[(x-1)^2 + x^2\right] dx$$

$$= \left[\frac{(x-1)^3}{3} + \frac{x^3}{3}\right]_{x=1}^{x=0}$$

$$= \left(-\frac{1}{3} + 0\right) - \left(0 + \frac{1}{3}\right)$$

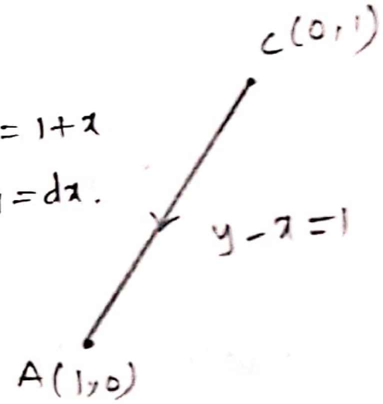
$$\int_{BC} y^2 dx - x^2 dy = -\frac{2}{3} \quad \text{——} (3)$$

Case (iii) Along the line CA (or) To evaluate $\int_{CA} y^2 dx - x^2 dy$.

We have C (0, 1) A (−1, 0)

Equation of the line CA is $y - x = 1 \Rightarrow y = 1 + x$

$dy = dx$.



x varies from 0 to −1

∴ x limits $x = 0, x = -1$.

$$\int_{CA} y^2 dx - x^2 dy = \int_{CA} (x+1)^2 dx - x^2 dx.$$

$$= \int_{x=0}^{x=-1} \left[(x+1)^2 - x^2\right] dx$$

$$= \left[\frac{(x+1)^3}{3} - \frac{x^3}{3}\right]_{x=0}^{x=-1}$$

$$= \left(0 + \frac{1}{3}\right) - \left(\frac{1}{3} - 0\right)$$

$$\int_{CA} y^2 dx - x^2 dy = 0 \quad \text{——} (4)$$

Sub. (2), (3) and (4) in (1), we get

$$\int_C y^2 dx - x^2 dy = 0 - \frac{2}{3} + 0$$

$$= -\frac{2}{3}$$

→ If $\bar{F} = (x^2+y^2)i - 2xy\,j$ evaluate $\oint_c \bar{F}\cdot d\bar{r}$ where curve c is the rectangle in xy-plane bounded by $x=0,\ x=a,\ y=0,\ y=b$.

Sol: Given that $\bar{F} = (x^2+y^2)i - 2xy\,j$

We have to find $\oint_c \bar{F}\cdot d\bar{r}$

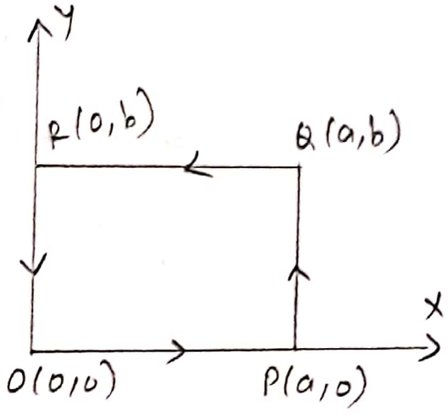
Let $\bar{r} = xi+yj+zk$

$\bar{r} = xi+yj$ [$\because$ In xy-plane $z=0$]

$d\bar{r} = dx\,i + dy\,j$

$\bar{F}\cdot d\bar{r} = [(x^2+y^2)i - 2xy\,j]\,[dx\,i + dy\,j]$

$\bar{F}\cdot d\bar{r} = (x^2+y^2)\,dx - 2xy\,dy$ —— ①



The curve c is the rectangle bounded by $x=0\ x=a,\ y=0,\ y=b$.

The vertices of the rectangle are $O(0,0)\ P(a,0)\ Q(a,b)\ R(0,b)$.

$$\oint \bar{F}\cdot d\bar{r} = \int_{OP} \bar{F}\cdot d\bar{r} + \int_{PQ} \bar{F}\cdot d\bar{r} + \int_{QR} \bar{F}\cdot d\bar{r} + \int_{RO} \bar{F}\cdot d\bar{r} \quad \text{—— ②}$$

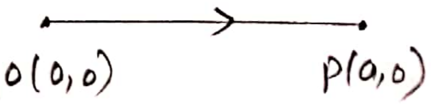
Case (i):- To evaluate $\int_{OP} \bar{F}\cdot d\bar{r}$ (or) Along the line OP :-

We have $O(0,0)\ P(a,0)$

Here $y=0 \Rightarrow dy=0$.

x varies from 0 to a.

$\therefore$ x limits are $x=0,\ x=a$.



From ①, $\bar{F}\cdot d\bar{r} = x^2\,dx$.

$$\int_{OP} \bar{F}\cdot d\bar{r} = \int_{OP} x^2\,dx$$

$$= \int_{x=0}^{x=a} x^2\,dx$$

$$= \left[\frac{x^3}{3}\right]_{x=0}^{x=a}$$

$$= \frac{a^3}{3}.$$

Case (ii) :- To evaluate $\int_{PQ} \bar{F}\cdot d\bar{r}$ (or) Along the line PQ :

We have $P(a,0)$ $Q(a,b)$

Here $x = a \Rightarrow dx = 0$.

y varies from 0 to b.

$\therefore$ y limits are $y=0,\ y=b$.

From (1), $\bar{F}\cdot d\bar{r} = -2ay\,dy$

$$\int_{PQ} \bar{F}\cdot d\bar{r} = \int_{PQ} -2ay\,dy$$

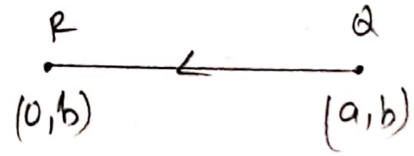
$$= \int_{y=0}^{y=b} -2ay\,dy$$

$$= \left[-2a\cdot\frac{y^2}{2}\right]_{y=0}^{y=b}$$

$$= -ab^2 \quad \text{—— (4)}$$

Case(iii) : To evaluate $\int_{QR} \bar{F}\cdot d\bar{r}$ (or) Along the line QR .

We have $Q(a,b)$ $R(0,b)$



Here $y = b \Rightarrow dy = 0$

x varies from a to a.

$\therefore$ x limits are $x = a,\ x = 0$.

From (1). $\bar{F}\cdot d\bar{r} = (x^2+b^2)\,dx$.

$$\int_{QR} \bar{F}\cdot d\bar{r} = \int_{QR} (x^2+b^2)\,dx$$

$$= \int_{x=a}^{x=0} (x^2+b^2)\,dx$$

$$= \left[\frac{x^3}{3} + b^2 x\right]_{x=a}^{x=0}$$

$$= 0 - \left(\frac{a^3}{3} + ab^2\right)$$

$$= -\frac{a^3}{3} - ab^2 \quad \text{—— (5)}$$

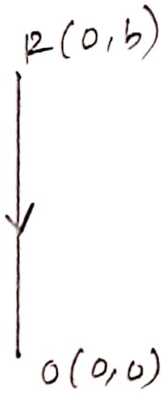
Case (iv) To evaluate $\int_{RO} \bar{F}\cdot d\bar{r}$ (or) Along the line RO :-

We have $R(0,b)$ $O(0,0)$

Here $x=0 \Rightarrow dx=0$.

y varies from b to 0.

$\therefore$ y limits $y=b$, $y=0$.



From (1), $\bar{F}\cdot d\bar{r} = 0$

$$\int_{RO} \bar{F}\cdot d\bar{r} = 0 \quad \text{——} (6)$$

Sub (3) (4), (5) and (6) in (2), we get

$$\oint_C \bar{F}\cdot d\bar{r} = \frac{a^3}{3} - ab^2 - \frac{a^3}{3} - ab^2$$

$$\oint_C \bar{F}\cdot d\bar{r} = -2ab^2$$

→ Evaluate the line integral $\int_C [(x^2+xy)\,dx + (x^2+y^2)\,dy]$ where c is the square formed by the lines $x=\pm 1$ and $y=\pm 1$

Ans: 0.

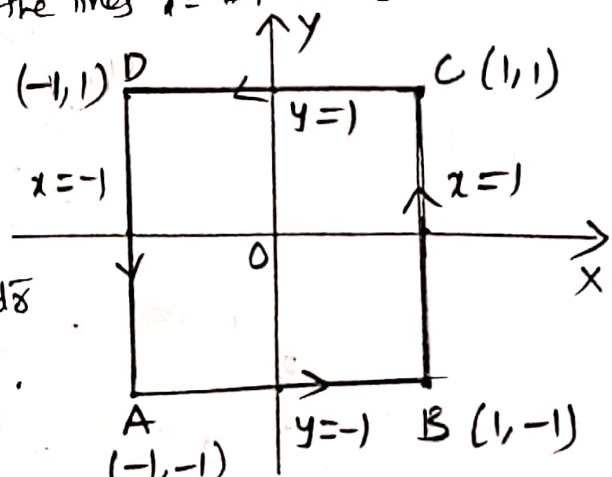
Sol: $I = \int_C (x^2+xy)\,dx + (x^2+y^2)\,dy$

Let $\bar{F}\cdot d\bar{r} = (x^2+xy)\,dx + (x^2+y^2)\,dy$ —— (1)

Given that c is the square formed by the lines $x=\pm 1$ and $y=\pm 1$.

The vertices of the square are

$A(-1,-1)$ $B(1,-1)$ $C(1,1)$ $D(-1,1)$



$$\int_C \bar{F}\cdot d\bar{r} = \int_{AB} \bar{F}\cdot d\bar{r} + \int_{BC} \bar{F}\cdot d\bar{r} + \int_{CD} \bar{F}\cdot d\bar{r} + \int_{DA} \bar{F}\cdot d\bar{r}$$

—— (2).

Case (i) To evaluate $\int_{AB} \bar{F}\cdot d\bar{r}$ (or) Along the line AB :-

We have $A(-1,-1)$ $B(1,-1)$

Here $y=-1 \Rightarrow dy=0$

x varies from -1 to 1

x limits $x=-1$, $x=1$

From (1), $\bar{F}\cdot d\bar{r} = (x^2-x)\,dx$ $[\because y=-1, dy=0]$

$$\int_{AB} \bar{F}\cdot d\bar{r} = \int_{AB} (x^2 - x)\,dx$$

$$= \int_{x=-1}^{x=1} (x^2 - x)\,dx$$

$$= \left[\frac{x^3}{3} - \frac{x^2}{2}\right]_{x=-1}^{x=1}$$

$$= \left(\frac{1}{3} - \frac{1}{2}\right) - \left(\frac{-1}{3} - \frac{1}{2}\right)$$

$$\int_{AB} \bar{F}\cdot d\bar{r} = \frac{2}{3} \quad \text{——— (3)}$$

<u>Case (ii)</u> <u>To evaluate</u> $\int_{BC} \bar{F}\cdot d\bar{r}$ <u>(or) Along the line BC</u> :—

We have $B(1,-1)$ $C(1,1)$.

Here $x = 1 \implies dx = 0$.

y varies from -1 to 1.

$\therefore$ y limits $y = -1$, $y = 1$.

From (1) $\bar{F}\cdot d\bar{r} = (y^2 + 1)\,dy$

$$\int_{BC} \bar{F}\cdot d\bar{r} = \int_{BC} (y^2 + 1)\,dy$$

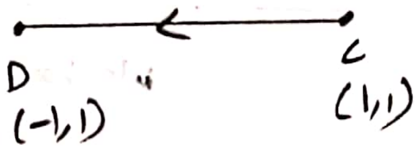
$$= \int_{y=-1}^{y=1} (y^2 + 1)\,dy$$

$$= \left[\frac{y^3}{3} + y\right]_{y=-1}^{y=1} = \left(\frac{1}{3} + 1\right) - \left(\frac{-1}{3} - 1\right).$$

$$\int_{BC} \bar{F}\cdot d\bar{r} = \frac{8}{3} \quad \text{——— (4)}$$

<u>Case (iii)</u> <u>To evaluate</u> $\int_{CD} \bar{F}\cdot d\bar{r}$ <u>(or) Along the line CD</u> :—

We have $C(1,1)$ $D(-1,1)$



Here $y = 1 \implies dy = 0$

x varies from -1 to 1.

$\therefore$ x limits $x = -1$, $x = 1$.

From (1). $\bar{F}\cdot d\bar{r} = (x^2 + x)\,dx$.

$$\int_{CD} \bar{F}\cdot d\bar{r} = \int_{CD} (x^2+x)\,dx$$

$$= \int_{x=-1}^{x=1} (x^2+x)\,dx$$

$$= \left[\frac{x^3}{3}+\frac{x^2}{2}\right]_{x=-1}^{x=1} = \left(\frac{1}{3}+\frac{1}{2}\right)-\left(\frac{-1}{3}+\frac{1}{2}\right)$$

$$\int_{CD} \bar{F}\cdot d\bar{r} = -\frac{2}{3} \quad \text{—— (5)}$$

Case (iv) To evaluate $\int_{DA} \bar{F}\cdot d\bar{r}$ (or) Along the line DA :-

We have $D(-1,1)$ $A(-1,-1)$

Here $x=-1 \Rightarrow dx=0$.

y varies from 1 to -1.

$\therefore$ y limits $y=1$, $y=-1$

From (1), $\bar{F}\cdot d\bar{r} = (y^2+1)\,dy$.

$$\int_{DA} \bar{F}\cdot d\bar{r} = \int_{DA} (y^2+1)\,dy$$

$$= \int_{y=1}^{y=-1} (y^2+1)\,dy$$

$$= \left[\frac{y^3}{3}+y\right]_{y=1}^{y=-1}$$

$$= \left(\frac{-1}{3}-1\right)-\left(\frac{1}{3}+1\right)$$

$$\int_{DA} \bar{F}\cdot d\bar{r} = -\frac{8}{3} \quad \text{—— (6)}$$

Sub. (3) (4) (5) and (6) in (2), we get

$$\int_{C} \bar{F}\cdot d\bar{r} = \frac{2}{3}+\frac{8}{3}-\frac{2}{3}-\frac{8}{3}$$

$$\int_{C} \bar{F}\cdot d\bar{r} = 0.$$

Green's Theorem in a plane :-

(Transformation Between Line Integral and Double Integral)

If R is a closed region in xy-plane bounded by a simple closed curve C and if M and N are continuous functions of x and y having continuous derivatives in R, then $\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

where C is traversed in the positive (anti clockwise) direction

Note :-

(i) Green's theorem converts a line integral around a closed curve into a double integral and is a special case of Stoke's theorem.

(ii) Green's theorem in a vector notation.

Let $\bar{F} = M\hat{i} + N\hat{j}$ and $\bar{r} = x\hat{i} + y\hat{j}$

$\bar{F}\cdot d\bar{r} = M\,dx + N\,dy$

$$\text{Curl}\,\bar{F} = \nabla \times \bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ M & N & 0 \end{vmatrix} = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\hat{k}.$$

$$\therefore \oint_C \bar{F}\cdot d\bar{r} = \iint_R \text{Curl}\,\bar{F}\cdot\bar{n}\,dR$$

where $\bar{n} = \hat{k}$ for xy plane and $dR = dx\,dy$ and $\text{Curl}\,\bar{F}\cdot\bar{n} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$.

(iii) Area of the plane region R bounded by a simple closed curve C.

Let $N = x$ and $M = -y$

Then $\oint_C x\,dy - y\,dx = \iint_R (1+1)\,dx\,dy = 2\iint_R dx\,dy$

Hence $A = \frac{1}{2}\oint_C x\,dy - y\,dx$.

1) Verify Green's theorem in plane for $\oint_C (3x^2-8y^2)\,dx + (4y-6xy)\,dy$ where C is the region bounded by $y=\sqrt{x}$ and $y=x^2$.

Sol:- Given that the integral $I = \oint_C (3x^2-8y^2)\,dx + (4y-6xy)\,dy$ — ①

WKT Green's Theorem in a plane.

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy \quad — ②$$

Compare ① with L.H.S of ②, we get

$$M = 3x^2 - 8y^2 \qquad N = 4y - 6xy.$$

Given that the region bounded by the curves $\underbrace{y=\sqrt{x}}_{③}$ and $\underbrace{y=x^2}_{④}$

solving ③ and ④, we get

$$y=\sqrt{x}, \quad y=x^2$$

$$x^2 = \sqrt{x} \Rightarrow x^4 = x$$

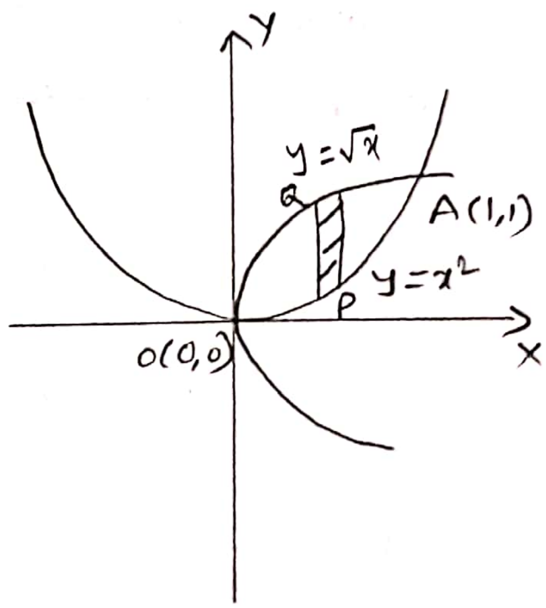
$$x(x^3-1) = 0$$

$$x = 0, 1$$

when $x=0$, $y=0$.

when $x=1$, $y=1$

The points of intersection of the curves ③ and ④ is. $O(0,0)$ $A(1,1)$.



To evaluate. $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

$$M = 3x^2 - 8y^2 \qquad N = 4y - 6xy$$

$$\frac{\partial M}{\partial y} = -16y \qquad \frac{\partial N}{\partial x} = -6y$$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 10y.$$

Draw a vertical strip PQ in the region.

We have to fix x first.

In the region x varies from 0 to 1.

$\therefore$ x limits are $x=0$, $x=1$.

For each x, y varies a point P on parabola $y = x^2$ to a point Q on the parabola $y = \sqrt{x}$.

$\therefore$ y limits are $y = x^2$ and $y = \sqrt{x}$.

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \int_{x=0}^{x=1} \int_{y=x^2}^{y=\sqrt{x}} 10y \, dy \, dx$$

$$= \int_{x=0}^{x=1} \left[10 \cdot \frac{y^2}{2}\right]_{y=x^2}^{y=\sqrt{x}} dx$$

$$= 5 \int_{x=0}^{x=1} \left[(\sqrt{x})^2 - (x^2)^2\right] dx$$

$$= 5 \int_{x=0}^{x=1} (x - x^4)\, dx = 5\left[\frac{x^2}{2} - \frac{x^5}{5}\right]_{x=0}^{x=1}$$

$$= 5\left(\frac{1}{2} - \frac{1}{5}\right)$$

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \frac{3}{2} \quad \text{--- (5)}$$

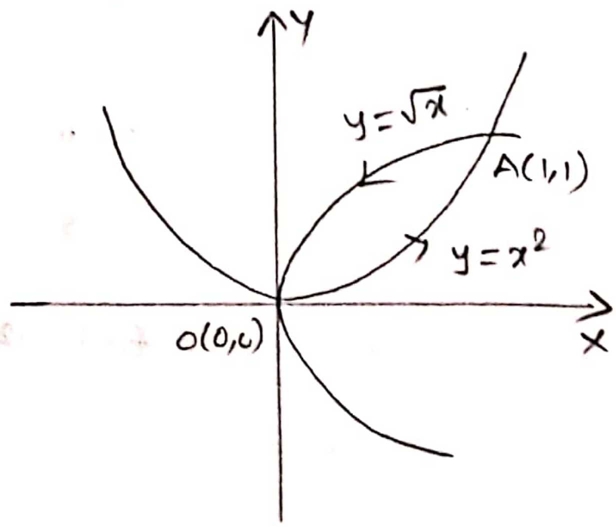
To evaluate $\oint_C M\,dx + N\,dy$:-

The region bounded by the curves $y = x^2$ and $y = \sqrt{x}$.

To evaluate the line integral $\oint_C M\,dx + N\,dy$.

We can write $\oint_C M\,dx + N\,dy = \int_{OA} M\,dx + N\,dy + \int_{AO} M\,dx + N\,dy$ --- (6)

Case (i) :- To evaluate $\int_{OA} M\,dx + N\,dy$ (OR) Along the curve $y = x^2$.



We have $y = x^2$

$dy = 2x\,dx$.

We have O(0,0) A(1,1)

Here x varies from 0 to 1.

$\therefore$ x limits are $x=0$, $x=1$.

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$$\int_{OA} Mdx + Ndy = (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy \quad [\because y = x^2,\ dy = 2x\,dx]$$

$$= (3x^2 - 8(x^2)^2)dx + (4x^2 - 6x \cdot x^2)(2x\,dx)$$

$$Mdx + Ndy = (3x^2 + 8x^3 - 20x^4)\,dx.$$

$$\int_{OA} Mdx + Ndy = \int_{OA} (3x^2 + 8x^3 - 20x^4)\,dx$$

$$= \int_{x=0}^{x=1} (3x^2 + 8x^3 - 20x^4)\,dx$$

$$= \left[3 \cdot \frac{x^3}{3} + 8 \cdot \frac{x^4}{4} - 20 \cdot \frac{x^5}{5}\right]_{x=0}^{x=1}$$

$$= \left[x^3 + 2x^4 - 4x^5\right]_{x=0}^{x=1}$$

$$= (1 + 2 - 4) - 0$$

$$\int_{OA} Mdx + Ndy = -1 \quad \text{—— (7)}$$

<u>case (ii)</u> :- To evaluate $\int_{AO} Mdx + Ndy$ (OR) Along the curve $y = \sqrt{x}$.

We have $y = \sqrt{x}$

i.e $x = y^2$

$dx = 2y\,dy$

We have ~~O(0,0)~~ $A(1,1)$ $O(0,0)$

Here y varies from 1 to 0

$\therefore$ y limits are $y = 1$, $y = 0$

$$Mdx + Ndy = (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$$

$$= (3y^4 - 8y^2)(2y\,dy) + (4y - 6y \cdot y^2)\,dy \quad \left[\because x = y^2,\ dx = 2y\,dy\right]$$

$$= \left[6y^5 - 16y^3 + 4y - 6y^3\right]dy$$

$$Mdx + Ndy = (6y^5 + 4y - 22y^3)\,dy$$

$$Mdx + Ndy = (6y^5 + 4y - 22y^3)\,dy$$

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$$\int_{AO} M\,dx + N\,dy = \int_{AO} (6y^5 + 4y - 22y^3)\,dy$$

$$= \int_{y=1}^{y=0} (6y^5 + 4y - 22y^3)\,dy$$

$$= \left[6 \cdot \frac{y^6}{6} + 4\frac{y^2}{2} - 22\frac{y^4}{4}\right]_{y=1}^{y=0}$$

$$= \left[y^6 + 2y^2 - \frac{11}{2}y^4\right]_{y=1}^{y=0}$$

$$= 0 - \left(1 + 2 - \frac{11}{2}\right)$$

$$\int_{AO} M\,dx + N\,dy = \frac{5}{2} \quad \text{—— (8)}$$

sub. (7) and (8) in (6), we get

$$\oint_C M\,dx + N\,dy = -1 + \frac{5}{2} = \frac{3}{2} \quad \text{—— (9)}$$

$\therefore$ From (5) and (9),

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

2) Verify Green's theorem for $\oint_C (2xy - x^2)\,dx + (x^2 + y^2)\,dy$ where c is the closed curve in xy - plane bounded by the curves $y = x^2$ and $y^2 = x$. Ans:- $\frac{1}{30}$.

Sol:- Given that $I = \oint_C (2xy - x^2)\,dx + (x + y^2)\,dy$ —— (1)

WKT Green's Theorem $\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$ —— (2)

Compare (1) with L.H.S of (2), we get

$M = 2xy - x^2 \qquad N = x + y^2$

$\frac{\partial M}{\partial y} = 2x \qquad \frac{\partial N}{\partial x} = 1.$

The region bounded by the curves $y = x^2$ and $y^2 = x$.

Solving $y = x^2$, $y^2 = x$

$y^2 = x \Rightarrow x^4 = x$

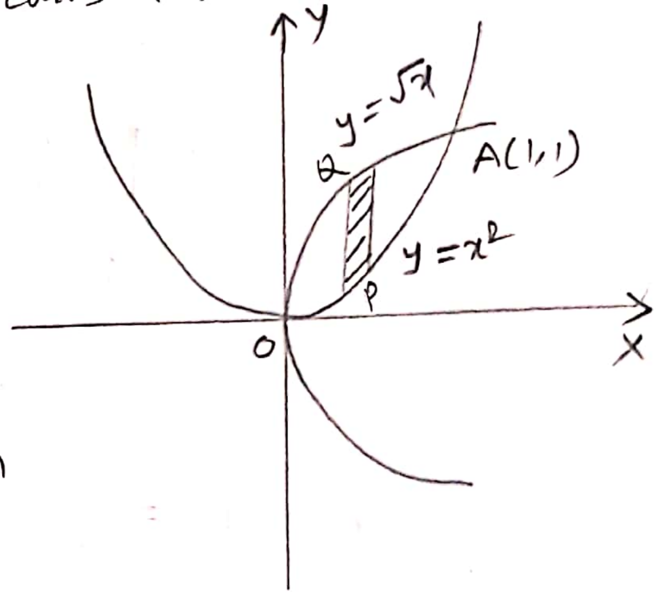
$x(x^3 - 1) = 0$

$x = 0, 1$

When $x=0,\ y=0$.

When $x=1,\ y=1$.

$\therefore$ The points of intersection of the curves $O(0,0)$ $A(1,1)$



To evaluate $\iint_R \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right) dx\,dy$

$M=2xy-x^2 \qquad N=x+y^2$

$\frac{\partial M}{\partial y}=2x \qquad \frac{\partial N}{\partial x}=1$

$\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}=1-2x.$

Draw a vertical strip PQ in the region

We have to fix x first.

In the region x varies from 0 to 1.

$\therefore$ x limits $x=0,\ x=1.$

For each x, y varies from a point P on $y=x^2$ to a point Q on $y^2=x$.

$\therefore$ y limits $y=x^2,\ y=\sqrt{x}$.

$$\iint_R \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right) dx\,dy = \int_{x=0}^{x=1}\int_{y=x^2}^{y=\sqrt{x}} (1-2x)\,dy\,dx.$$

$$= \int_{x=0}^{x=1} (1-2x)\,[y]_{y=x^2}^{y=\sqrt{x}}\,dx.$$

$$= \int_{x=0}^{x=1} (1-2x)(\sqrt{x}-x^2)\,dx$$

$$= \int_{x=0}^{x=1} \left(\sqrt{x}-2x^{3/2}-x^2+2x^3\right) dx.$$

$$= \left[\frac{2}{3}x^{3/2}-2\cdot\frac{2}{5}x^{5/2}-\frac{x^3}{3}+2\cdot\frac{x^4}{4}\right]_{x=0}^{x=1}$$

$$= \frac{2}{3}-\frac{4}{5}-\frac{1}{3}+\frac{1}{2}$$

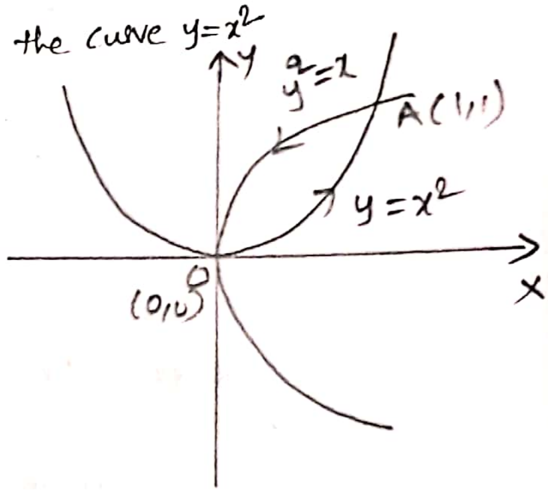
$$\iint_R \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right) dx\,dy = \frac{1}{30}$$

To evaluate. $\oint_C M\,dx + N\,dy$:-

To evaluate the line integral $\oint_C M\,dx + N\,dy$

We can write. $\oint_C M\,dx + N\,dy = \int_{OA} M\,dx + N\,dy + \int_{AO} M\,dx + N\,dy$ —— (3)

Case (i) :- To evaluate $\int_{OA} M\,dx + N\,dy$ (OR) Along the curve $y = x^2$



We have $y = x^2$

$dy = 2x\,dx$

We have $O(0,0)$ $A(1,1)$

x varies from 0 to 1.

$\therefore$ x limits $x = 0$, $x = 1$.

$M\,dx + N\,dy = (2xy - x^2)\,dx + (x + y^2)\,dy$

$= (2x^3 - x^2)\,dx + (x^2 + x^4)\,2x\,dx$

$M\,dx + N\,dy = (2x^3 + x^2 + 2x^5)\,dx$

$\int_{OA} M\,dx + N\,dy = \int_{OA} (2x^3 + x^2 + 2x^5)\,dx$

$= \int_{x=0}^{x=1} (2x^3 + x^2 + 2x^5)\,dx$

$= \left[\frac{x^4}{2} + \frac{x^3}{3} + \frac{x^6}{3}\right]_{x=0}^{x=1}$

$\int_{OA} M\,dx + N\,dy = \frac{7}{6}$ —— (4)

Case (ii) To evaluate $\int_{AO} M\,dx + N\,dy$ (OR) Along the curve $y^2 = x$.

We have $x = y^2$

$dx = 2y\,dy$

We have $A(1,1)$ $O(0,0)$

y varies from 1 to 0.

$\therefore$ y limits $y = 1$, $y = 0$.

$M\,dx + N\,dy = (2xy - x^2)\,dx + (x + y^2)\,dy$

$= (2y^3 - y^4)\,2y\,dy + (y^2 + y^2)\,dy$

$$Mdx + Ndy = (4y^4 - 2y^5 + 2y^2)dy$$

$$\int_{AO} M\,dx + N\,dy = \int_{AO} (4y^4 - 2y^5 + 2y^2)\,dy$$

$$= \int_{y=1}^{y=0} (4y^4 - 2y^5 + 2y^2)\,dy$$

$$= \left[4 \cdot \frac{y^5}{5} - 2\frac{y^6}{6} + 2\frac{y^3}{3}\right]_{y=1}^{y=0}$$

$$= \frac{4}{5} + \frac{1}{3} - \frac{2}{3}$$

$$\int_{AO} M\,dx + N\,dy = \frac{-17}{15} \quad \text{---} (5)$$

Sub (4) and (5) in (3), we get

$$\oint_C M\,dx + N\,dy = \frac{7}{6} - \frac{17}{15} = \frac{1}{30}$$

$$\therefore \oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$\therefore$ Green's theorem verified.

3) Verify Green's theorem for $\int_C [(xy+y^2)\,dx + x^2\,dy]$ where c is bounded by $y=x$ and $y=x^2$

Ans:- $-\frac{1}{20}$.

Sol:- Given that $I = \int_C [(xy+y^2)\,dx + x^2\,dy]$ —— ①

WKT Greens theorem

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy \quad \text{—— ②}$$

Compare ① with L.H.S ② we get

$M = xy + y^2 \qquad N = x^2.$

The region is bounded by $y=x$ —— ① and $y=x^2$ —— ②

Solving ① and ②,

$y = x^2$ and $y = x$

$x^2 = x$

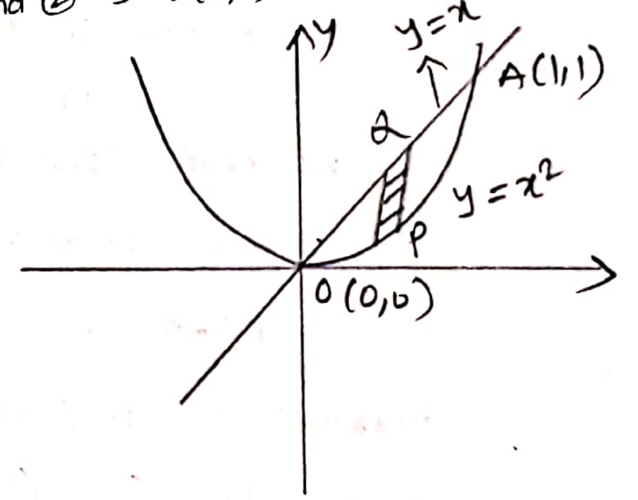
$x(x-1) = 0$

$x=0,\ x=1.$

When $x=0,\quad y=0$

When $x=1,\quad y=1.$

$\therefore$ The points of intersection of ① and ② is $O(0,0)$ and $A(1,1)$



To evaluate $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$:-

$M = xy + y^2 \qquad N = x^2$

$\frac{\partial M}{\partial y} = x + 2y \qquad \frac{\partial N}{\partial x} = 2x$

$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = x - 2y$

Draw a vertical strip PQ in the region

We have to fix x first

In the region x varies from 0 to 1.

$\therefore$ x limits $x=0,\ x=1.$

For each x, y varies from a point P on the parabola $y=x^2$ to a point Q on the line $y=x$. $\quad\therefore$ y limits $y=x^2,\ y=x.$

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \iint_R (x-2y)\,dy\,dx$$

$$= \int_{x=0}^{x=1} \int_{y=x^2}^{y=x} (x-2y)\,dy\,dx$$

$$= \int_{x=0}^{x=1} \left[xy - 2\cdot\frac{y^2}{2}\right]_{y=x^2}^{y=x} dx$$

$$= \int_{x=0}^{x=1} (x^4 - x^3)\,dx$$

$$= \left[\frac{x^5}{5} - \frac{x^4}{4}\right]_{x=0}^{x=1}$$

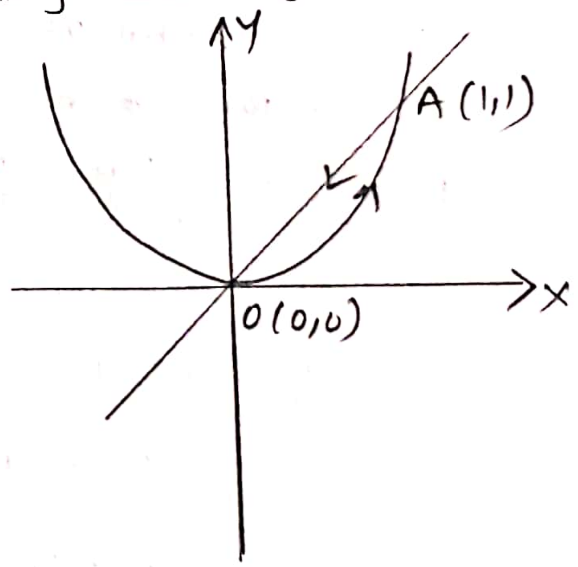
$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \frac{-1}{20} \quad \text{--- (3)}$$

<u>To evaluate</u> $\oint_C$ <u>$M\,dx + N\,dy$</u> :-

To evaluate the line integral $\oint_C M\,dx + N\,dy$

We can write $\oint_C M\,dx + N\,dy = \int_{OA} M\,dx + N\,dy + \int_{AO} M\,dx + N\,dy$ --- (4)

<u>Case (i)</u> To evaluate $\int_{OA} M\,dx + N\,dy$ (or) Along the curve $y = x^2$



We have $y = x^2$

$dy = 2x\,dx$.

We have $O(0,0)$ $A(1,1)$

x varies from 0 to 1

$\therefore$ x limits $x=0, x=1$.

$M\,dx + N\,dy = (xy + y^2)\,dx + x^2\,dy$

$M\,dx + N\,dy = (x^3 + x^4 + 2x^3)\,dx$

$= (3x^3 + x^4)\,dx$

$$\int_{OA} M\,dx + N\,dy = \int_{OA} (3x^3 + x^4)\,dx$$

$$= \int_{x=0}^{x=1} (3x^3 + x^4)\,dx$$

$$\int_{OA} M\,dx + N\,dy = \left[3\frac{x^4}{4} + \frac{x^5}{5}\right]_{x=0}^{x=1}$$

$$= \frac{3}{4} + \frac{1}{5}$$

$$= \frac{19}{20} \quad \text{—— (5)}$$

<u>Case (ii)</u> To evaluate $\int_{AO} M\,dx + N\,dy$ (or) Along the line $y = x$.

We have $y = x \Rightarrow dy = dx$

We have $A(1,1)$ $O(0,0)$

$\therefore x$ varies from 1 to 0

$\therefore$ x limits $x = 1, x = 0$

$$M\,dx + N\,dy = (xy + y^2)\,dx + x^2\,dy$$

$$= 3x^2\,dx$$

$$\int_{AO} M\,dx + N\,dy = \int_{AO} 3x^2\,dx$$

$$= \int_{x=1}^{x=0} 3x^2\,dx$$

$$= \left[3 \cdot \frac{x^3}{3}\right]_{x=1}^{x=0}$$

$$\int_{AO} M\,dx + N\,dy = 3\left(-\frac{1}{3}\right) = -1 \quad \text{—— (6)}$$

sub (5) & (6) in (4), we get

$$\oint_C M\,dx + N\,dy = \frac{19}{20} - 1 = \frac{-1}{20}$$

$$\therefore \oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$\therefore$ Green's theorem verified.

→ Verify Green's theorem for $\oint_c (y-\sin x)\,dx + \cos x\,dy$ where c is the triangle enclosed by the lines $y=0$, $x=\frac{\pi}{2}$, $\pi y = 2x$. (OR)

Verify Green's theorem for $\oint_c (y-\sin x)\,dx + \cos x\,dy$ where c is the boundary of the triangle in xy-plane whose vertices are $(0,0)$ $(\frac{\pi}{2},0)$ and $(\frac{\pi}{2},1)$ traversed in the positive direction.

Sol:- Given that $I = \oint_c (y-\sin x)\,dx + \cos x\,dy$ —— ①

W.K.T Green's theorem in a plane.

$$\oint_c M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy \quad —— ②$$

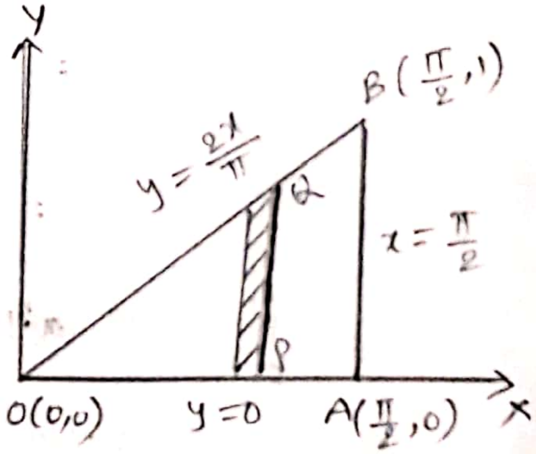
Compare ① with L.H.S of ②, we get

Here $M = y-\sin x$ $N = \cos x$.

Given that c is the triangle enclosed by the lines $y=0$, $x=\frac{\pi}{2}$, $y=\frac{2x}{\pi}$

The vertices of the triangle are $O(0,0)$ $A(\frac{\pi}{2},0)$ $B(\frac{\pi}{2},1)$.

<u>To evaluate $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$:-</u>



$M = y-\sin x$ $N = \cos x$

$\frac{\partial M}{\partial y} = 1$ $\frac{\partial N}{\partial x} = -\sin x$.

$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = -\sin x - 1$.

Draw a vertical strip PQ in the region

We have to fix x first

In the region, x varies from 0 to $\frac{\pi}{2}$.

∴ x limits are $x=0$, $x=\frac{\pi}{2}$

For each x, y varies from a point P on x-axis $(y=0)$ to a point Q on the line $y=\frac{2x}{\pi}$.

∴ y limits are $y=0$, $y=\frac{2x}{\pi}$.

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \iint_R (-1-\sin x)\,dx\,dy$$

$$= \int_{x=0}^{x=\pi/2} \int_{y=0}^{y=\frac{2x}{\pi}} (-1-\sin x)\,dy\,dx.$$

$$= \int_{x=0}^{x=\frac{\pi}{2}} (-1-\sin x)\,[y]_{y=0}^{y=\frac{2x}{\pi}}\,dx$$

$$= -\int_{x=0}^{x=\pi/2} (1+\sin x)\,\frac{2x}{\pi}\,dx$$

$$= -\frac{2}{\pi}\int_{x=0}^{x=\pi/2} (x + x\sin x)\,dx$$

$$= -\frac{2}{\pi}\left[\int_{x=0}^{x=\pi/2} x\,dx + \int_{x=0}^{x=\pi/2} x\sin x\,dx\right]$$

$$= -\frac{2}{\pi}\left[\left[\frac{x^2}{2}\right]_{x=0}^{x=\pi/2} + \left[x(-\cos x) - 1.(-\sin x)\right]_{x=0}^{x=\frac{\pi}{2}}\right]$$

$$= -\frac{2}{\pi}\left[\left[\frac{\pi^2}{8} - 0\right] + \left\{\left(-\frac{\pi}{2}\cos\frac{\pi}{2} + \sin\frac{\pi}{2}\right) - 0\right\}\right]$$

$$= -\frac{2}{\pi}\left(\frac{\pi^2}{8} + 1\right)$$

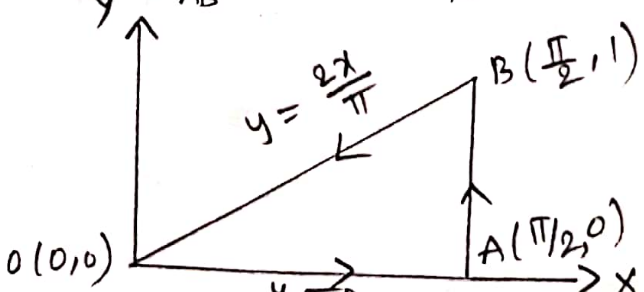
$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = -\left(\frac{\pi}{4} + \frac{2}{\pi}\right) \quad \text{--- (3)}$$

To evaluate $\oint_C M\,dx + N\,dy$:-

The region is $\triangle OAB$.

To evaluate the line integral $\oint_C M\,dx + N\,dy$. ——— (4)

We can write $\oint_C M\,dx + N\,dy = \int_{OA} M\,dx + N\,dy + \int_{AB} M\,dx + N\,dy + \int_{BO} M\,dx + N\,dy$



Case (i) :- To evaluate $\int_{OA} Mdx + Ndy$ (OR) Along the line OA :-

We have $O(0,0)$ $A(\frac{\pi}{2}, 0)$

Here $y = 0 \Rightarrow dy = 0$.

x varies from 0 to $\frac{\pi}{2}$.

$\therefore$ x limits are $x = 0$, $x = \frac{\pi}{2}$

$Mdx + Ndy = (y - \sin x)dx + \cos x \, dy$

$Mdx + Ndy = -\sin x \, dx$ $[\because y = 0, dy = 0]$

$$\int_{OA} Mdx + Ndy = \int_{OA} -\sin x \, dx$$

$$= \int_{x=0}^{x=\pi/2} -\sin x \, dx$$

$$= [\cos x]_{x=0}^{x=\pi/2} = \cos\frac{\pi}{2} - \cos 0.$$

$$\int_{OA} Mdx + Ndy = -1 \text{ --- } (5).$$

Case (ii) :- To evaluate $\int_{AB} Mdx + Ndy$ (OR) Along the line AB :-

We have $A(\frac{\pi}{2}, 0)$ $B(\frac{\pi}{2}, 1)$

Here $x = \frac{\pi}{2} \Rightarrow dx = 0$.

y varies from 0 to 1.

$\therefore$ y limits are $y = 0$, $y = 1$.

$Mdx + Ndy = (y - \sin x)dx + \cos x \, dy$

$Mdx + Ndy = (y - \sin\frac{\pi}{2})(0) + \cos\frac{\pi}{2} \cdot dy = 0$ $[\because x = \frac{\pi}{2}, dx = 0]$

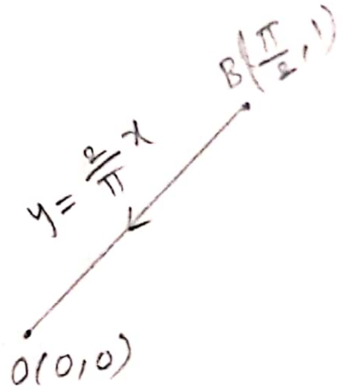
$$\int_{AB} Mdx + Ndy = \int_{AB} 0 = 0 \text{ --- } (6).$$

Case (iii) :- To evaluate $\int_{BO} Mdx + Ndy$ (OR) Along the line BO :

An equation of the line BO is $y = \frac{2x}{\pi}$

$dy = \frac{2}{\pi} dx$.

We have $B\left(\frac{\pi}{2}, 1\right)$ $O(0,0)$



x varies from $\frac{\pi}{2}$ to 0.

$\therefore$ x limits are $x = \frac{\pi}{2}, x = 0$.

$$Mdx + Ndy = (y - \sin x)dx + \cos x \, dy$$

$$= \left(\frac{2x}{\pi} - \sin x\right) dx + \frac{2}{\pi} \cos x \, dx$$

$$= \left[\frac{2x}{\pi} - \sin x + \frac{2}{\pi}\cos x\right] dx$$

$$\int_{BO} Mdx + Ndy = \int_{BO} \left[\frac{2x}{\pi} - \sin x + \frac{2}{\pi}\cos x\right] dx$$

$$= \int_{x=\frac{\pi}{2}}^{x=0} \left[\frac{2x}{\pi} - \sin x + \frac{2}{\pi}\cos x\right] dx$$

$$= \left[\frac{2}{\pi} \cdot \frac{x^2}{2} + \cos x + \frac{2}{\pi}\sin x\right]_{x=\frac{\pi}{2}}^{x=0}$$

$$= 1 - \left\{\frac{2}{\pi} \cdot \frac{1}{2} \frac{\pi^2}{4} + \cos\frac{\pi}{2} + \frac{2}{\pi}\sin\frac{\pi}{2}\right\}$$

$$\int_{BO} Mdx + Ndy = 1 - \frac{\pi}{4} - \frac{2}{\pi} \quad \text{—— (7)}$$

Sub. (5) (6) and (7) in (4), we get

$$\oint_C Mdx + Ndy = -1 + 0 + 1 - \frac{\pi}{4} - \frac{2}{\pi}$$

$$\oint_C Mdx + Ndy = -\left(\frac{\pi}{4} + \frac{2}{\pi}\right) \quad \text{—— (8)}$$

$\therefore$ From (4) and (8)

$$\therefore \oint_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy$$

$\therefore$ Green's theorem verified.

→ Verify Green's theorem for $\int_c (e^{-x} \sin y)\,dx + (e^{-x} \cos y)\,dy$ where c is the boundary of the rectangle whose vertices are $(0,0)$ $(\pi,0)$ $(\pi,\frac{\pi}{2})$ and $(0,\frac{\pi}{2})$ traversed in the positive direction.

Sol:- Given that $I = \int_c (e^{-x} \sin y)\,dx + (e^{-x}\cos y)\,dy$ — ①

WKT Green's theorem in a plane.

$$\oint_c M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy \quad \text{— ②}$$

Compare ① with L.H.S of ②, we get

$M = e^{-x} \sin y \qquad N = e^{-x}\cos y$.

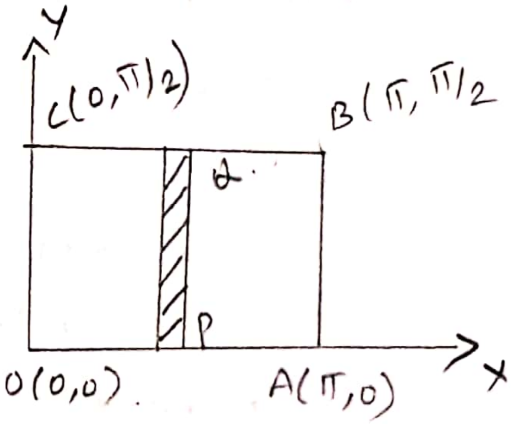
Given that c is the boundary of the rectangle whose vertices are $O(0,0)$, $A(\pi,0)$, $B(\pi,\frac{\pi}{2})$ and $C(0,\frac{\pi}{2})$.

To evaluate $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

$M = e^{-x} \sin y \qquad N = e^{-x} \cos y$

$\frac{\partial M}{\partial y} = e^{-x} \cos y \qquad \frac{\partial N}{\partial x} = -e^{-x}\cos y$

$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = -e^{-x}\cos y - e^{-x}\cos y = -2e^{-x}\cos y$



Draw a vertical strip PQ in the region.

We have to fix x first.

In the region x varies from 0 to π ∴ x limits are $x=0$, $x=\pi$

For each x, y varies from a point P on x-axis $(y=0)$ to a point Q on the line $y=\frac{\pi}{2}$.

∴ y limits are $y=0$, $y=\frac{\pi}{2}$.

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \iint_R -2e^{-x}\cos y\,dx\,dy$$

$$= \int_{y=0}^{y=\pi/2} \int_{x=0}^{x=\pi} -2e^{-x}\cos y\,dx\,dy$$

$$= -2\int_{y=0}^{y=\pi/2} \left[\frac{e^{-x}}{-1}\right]_{x=0}^{x=\pi} \cos y\,dy$$

$$= 2\int_{y=0}^{y=\pi/2} (e^{-\pi}-1)\cos y\,dy$$

$$= 2(e^{-\pi}-1)\left[\sin y\right]_{y=0}^{y=\pi/2}$$

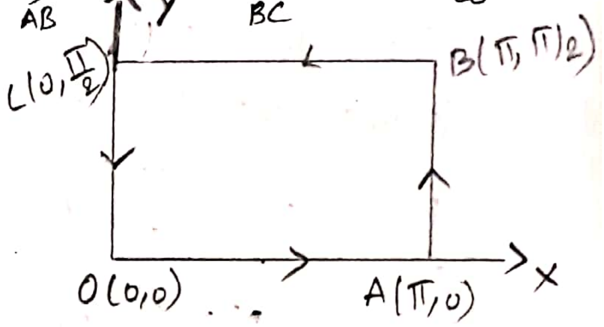
$$\iint_R \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dx\,dy = 2(e^{-\pi}-1) \quad \text{—— (3)}$$

<u>To evaluate $\oint_C M\,dx + N\,dy$:-</u>

The region is rectangle OABC.

To evaluate the line integral $\oint M\,dx + N\,dy$.

We can write $\oint_C M\,dx+N\,dy = \int_{OA} M\,dx+N\,dy + \int_{AB} M\,dx+N\,dy + \int_{BC} M\,dx+N\,dy + \int_{CO} M\,dx+N\,dy$ —— (4)



<u>Case(i)</u>:- To evaluate $\int_{OA} M\,dx + N\,dy$ (OR)

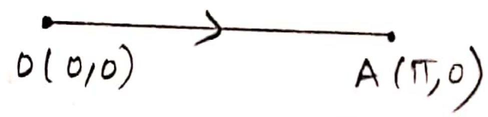
Along the line OA :-

We have $O(0,0)$ $A(\pi,0)$.

Here $y=0 \Rightarrow dy=0$.

x varies from 0 to π.

$\therefore$ x limits are $x=0$, $x=\pi$



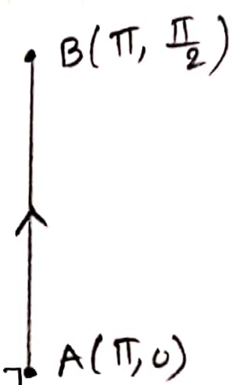
$M\,dx+N\,dy = e^{-x}\sin y\,dx + e^{-x}\cos y\,dy$. $[\because y=0,\ dy=0]$

$M\,dx+N\,dy = e^{-x}\sin(0)\,dx + e^{-x}\cos(0).0$

$M\,dx+N\,dy = 0$.

$$\int_{OA} M\,dx+N\,dy = \int_{OA} 0. = 0. \quad \text{—— (5)}$$

<u>Case(ii)</u>: To evaluate $\int_{AB} M\,dx + N\,dy$ (OR) Along the line AB :-



We have $A(\pi,0)$ $B(\pi,\frac{\pi}{2})$.

Here $x=\pi \Rightarrow dx=0$.

y varies from 0 to $\frac{\pi}{2}$.

$\therefore$ y limits are $y=0$, $y=\frac{\pi}{2}$.

$M\,dx+N\,dy = e^{-x}\sin y\,dx + e^{-x}\cos y\,dy$ $[\because x=\pi, dx=0]$

$M\,dx+N\,dy = e^{-\pi}\sin y\,(0) + e^{-\pi}\cos y\,.dy = +e^{-\pi}\cos y\,dy$

$$\int_{AB} M\,dx + N\,dy = \int_{AB} +e^{-\pi}\cos y\,dy$$

$$= \int_{y=0}^{y=\frac{\pi}{2}} +e^{-\pi}\cos y\,dy$$

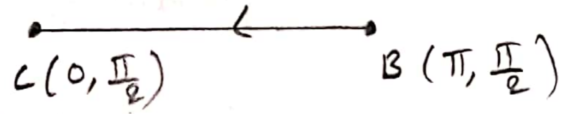
$$= \left[+e^{-\pi}\sin y\right]_{y=0}^{y=\frac{\pi}{2}}$$

$$= +e^{-\pi}\left[\sin\frac{\pi}{2} - \sin 0\right]$$

$$\int_{AB} M\,dx + N\,dy = +e^{-\pi} \quad \text{——— (6)}$$

Case (iii) To evaluate $\int_{BC} M\,dx + N\,dy$ (OR) Along the line BC :-

We have $B(\pi, \frac{\pi}{2})$ $C(0, \frac{\pi}{2})$



Here $y = \frac{\pi}{2} \Rightarrow dy = 0$.

x varies from π to 0.

$\therefore$ x limits $x = \pi$, $x = 0$.

$$M\,dx + N\,dy = e^{-x}\sin y\,dx + e^{-x}\cos y\,dy \quad \left[\because y = \frac{\pi}{2},\ dy = 0\right]$$

$$= e^{-x}\sin\left(\frac{\pi}{2}\right) dx + e^{-x}\cos\left(\frac{\pi}{2}\right)\cdot 0$$

$$M\,dx + N\,dy = e^{-x}\,dx$$

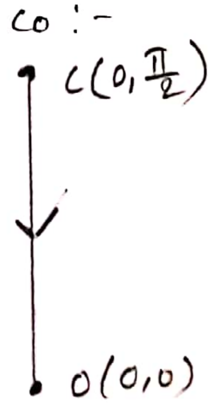
$$\int_{BC} M\,dx + N\,dy = \int_{BC} e^{-x}\,dx$$

$$= \int_{x=\pi}^{x=0} e^{-x}\,dx = \left[\frac{e^{-x}}{-1}\right]_{x=\pi}^{x=0}$$

$$= -\left[e^{0} - e^{-\pi}\right]$$

$$= e^{-\pi} - 1 \quad \text{——— (7)}.$$

Case (iv) To evaluate $\int_{CO} M\,dx + N\,dy$ (OR) Along the line CO :-



We have $C(0, \frac{\pi}{2})$ $O(0,0)$

Here $x = 0 \Rightarrow dx = 0$

y varies from $\frac{\pi}{2}$ to 0.

$\therefore$ y limits $y = \frac{\pi}{2}$, $y = 0$.

$$M\,dx + N\,dy = e^{-x}\sin y\, dx + e^{-x}\cos y\, dy \quad [\because x=0,\ dx=0]$$

$$= e^{-0}\sin y\,(0) + e^{-0}\cos y\, dy$$

$$M\,dx + N\,dy = \cos y\, dy$$

$$\int_{CO} M\,dx + N\,dy = \int_{CO} \cos y \,.\, dy$$

$$= \int_{y=\frac{\pi}{2}}^{y=0} \cos y\, dy$$

$$= \Big[\sin y\Big]_{y=\frac{\pi}{2}}^{y=0}$$

$$\int_{CO} M\,dx + N\,dy = -1 \quad \text{——} \ (8)$$

Sub. (5), (6) (7) and (8) in (4), we get

$$\oint_C M\,dx + N\,dy = e^{-\pi} + e^{-\pi} - 1 - 1$$

$$\oint_C M\,dx + N\,dy = 2(e^{-\pi} - 1) \quad \text{——} \ (9)$$

From (3) and (9)

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$\therefore$ Green's theorem verified.

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→ Verify Green's theorem in the plane for $\int_C (x^2 - xy^3)\,dx + (y^2 - 2xy)\,dy$ where c is a square with vertices $(0,0)$ $(2,0)$ $(0,2)$ $(2,2)$

Sol:- Given that $\int_C (x^2 - xy^3)\,dx + (y^2 - 2xy)\,dy$ —— ①

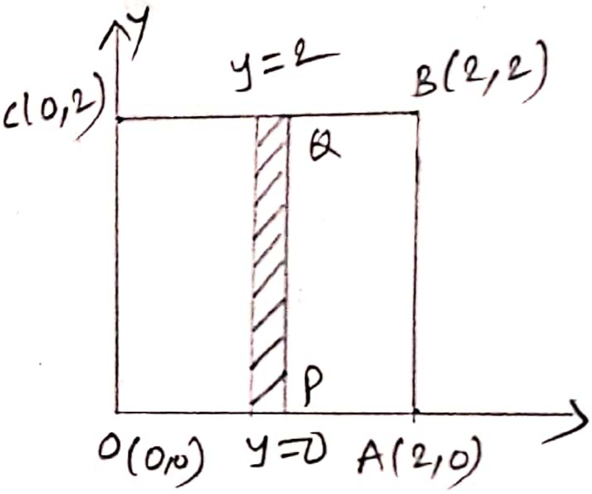
WKT Green's theorem in a plane

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy \text{ —— ②}$$

Compare ① with L.H.S of ②, we get

$$M = x^2 - xy^3 \qquad N = y^2 - 2xy$$

Given that c is the boundary of the square whose vertices are $O(0,0)$ $A(2,0)$ $B(2,2)$ $C(0,2)$.



To evaluate $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

$$M = x^2 - xy^3 \qquad N = y^2 - 2xy$$

$$\frac{\partial M}{\partial y} = -3xy^2 \qquad \frac{\partial N}{\partial x} = -2y$$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 3xy^2 - 2y.$$

Draw a vertical strip PQ in the region.

We have to fix x first

In the region x varies from 0 to 2

$\therefore$ x limits are $x=0$, $x=2$.

For each x, y varies from a point P on x-axis $(y=0)$ to a point Q on the line $y=2$.

$\therefore$ y limits are $y=0$, $y=2$.

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \iint_R (3xy^2 - 2y)\,dx\,dy$$

$$= \int_{y=0}^{y=2} \int_{x=0}^{x=2} (3xy^2 - 2y)\,dx\,dy$$

$$= \int_{y=0}^{y=2} \left[\frac{3x^2}{2} y^2 - 2xy\right]_{x=0}^{x=2} dy$$

$$= \int_{y=0}^{y=2} (6y^2 - 4y)\, dy$$

$$= \left[6\frac{y^3}{3} - 4\frac{y^2}{2} \right]_{y=0}^{y=2}$$

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\,dy = 8. \quad \text{---} \; (3)$$

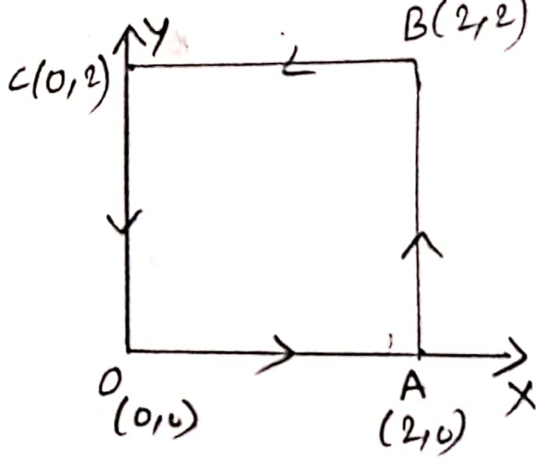
To evaluate $\oint_C M dx + N dy$:-

The region is square OABC.

To evaluate the line integral $\oint_C Mdx + Ndy$. —— (4)

We can write $\oint_C Mdx + Ndy = \int_{OA} Mdx + Ndy + \int_{AB} Mdx + Ndy + \int_{BC} Mdx + Ndy + \int_{CO} Mdx + Ndy$

Case(i) To evaluate $\int_{OA} Mdx + Ndy$ (or) Along the line OA :-



We have O(0,0) A(2,0)

Here $y = 0 \Rightarrow dy = 0$

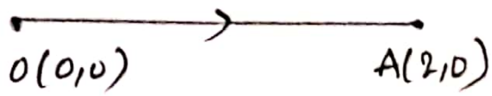
x varies from 0 to 2

$\therefore$ x limits $x = 0,\ x = 2$.

$Mdx + Ndy = (x^2 - xy^3)\,dx + (y^2 - 2xy)\,dy$

$Mdx + Ndy = x^2 dx \qquad [\because y = 0, dy = 0]$

$$\int_{OA} Mdx + Ndy = \int_{OA} x^2 dx$$



$$= \int_{x=0}^{x=2} x^2 dx$$

$$= \left[\frac{x^3}{3} \right]_{x=0}^{x=2}$$

$$\int_{OA} Mdx + Ndy = \frac{8}{3} \quad \text{---} \; (5)$$

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Case (ii) To evaluate $\int_{AB} M\,dx + N\,dy$ (OR) Along the line AB :-

We have $A(2,0)$ $B(2,2)$.

Here $x = 2 \Rightarrow dx = 0$

y varies from 0 to 2

$\therefore$ y limits $y = 0,\ y = 2$

$$M\,dx + N\,dy = (x^2 - xy^3)\,dx + (y^2 - 2xy)\,dy$$

$$M\,dx + N\,dy = (y^2 - 4y)\,dy \quad [\because x = 2,\ dx = 0]$$

$$\int_{AB} M\,dx + N\,dy = \int_{AB} (y^2 - 4y)\,dy$$

$$= \int_{y=0}^{y=2} (y^2 - 4y)\,dy$$

$$= \left[\frac{y^3}{3} - 4\frac{y^2}{2}\right]_{y=0}^{y=2}$$

$$= \frac{8}{3} - 8$$

$$\int_{AB} M\,dx + N\,dy = -\frac{16}{3} \quad \text{—— (6)}$$

Case (iii) To evaluate $\int_{BC} M\,dx + N\,dy$ (OR) Along the line BC :-

We have $B(2,2)$ $C(0,2)$

Here $y = 2 \Rightarrow dy = 0$

x varies from 2 to 0

$\therefore$ x limits $x = 2,\ x = 0$.

$$M\,dx + N\,dy = (x^2 - xy^3)\,dx + (y^2 - 2xy)\,dy$$

$$M\,dx + N\,dy = (x^2 - 8x)\,dx \quad [\because y = 2,\ dy = 0]$$

$$\int_{BC} M\,dx + N\,dy = \int_{BC} (x^2 - 8x)\,dx$$

$$= \int_{x=2}^{x=0} (x^2 - 8x)\,dx$$

$$= \left[\frac{x^3}{3} - 8\frac{x^2}{2}\right]_{x=2}^{x=0}$$

$$\int_{BC} M\,dx + N\,dy = +\frac{40}{3} \quad \text{—— (7)}$$

Case (iv) To evaluate $\int_{CO} Mdx + Ndy$ (OR) Along the line CO :-

We have $C(0,2)$ $O(0,0)$

Here $x = 0 \Rightarrow dx = 0$

y varies from 2 to 0.

$\therefore$ y limits $y = 2, y = 0$.

$$M dx + N dy = (x^2 - xy^3)\,dx + (y^2 - 2xy)\,dy$$

$$M dx + N dy = y^2\,dy$$

$$\int_{CO} Mdx + Ndy = \int_{CO} y^2\,dy$$

$$= \int_{y=2}^{y=0} y^2\,dy$$

$$= \left[\frac{y^3}{3}\right]_{y=2}^{y=0}$$

$$\int_{CO} Mdx + Ndy = -\frac{8}{3} \quad \text{——} (8)$$

Sub. (5) (6) (7) and (8) in (4), we get

$$\oint_C Mdx + Ndy = \frac{8}{3} - \frac{16}{3} + \frac{40}{3} - \frac{8}{3}$$

$$\oint_C Mdx + Ndy = 8$$

$$\therefore \oint_C Mdx + Ndy = \iint \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$\therefore$ Green's theorem verified.

Verify Green's theorem for $\oint_C (x^2 - \cosh y)\,dx + (y + \sin x)\,dy$ where C is the rectangle with vertices $(0,0)$ $(\pi,0)$ $(\pi,1)$ $(0,1)$.

Ans: $\pi(\cosh 1 - 1)$

Sol:- Given that $\oint_C (x^2 - \cosh y)\,dx + (y + \sin x)\,dy$ —— ①

W.k.t Green's theorem in a plane.

$$\oint M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy \quad \text{—— ②}$$

Compare ① with L.H.S of ②, we get

$M = x^2 - \cosh y \qquad N = y + \sin x$.

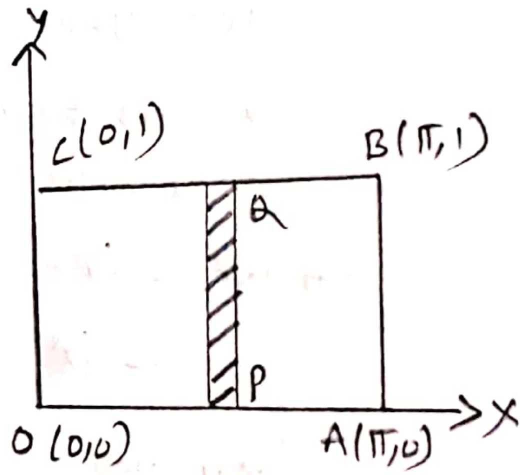
Given that C is the rectangle with vertices $O(0,0)$ $A(\pi,0)$ $B(\pi,1)$ and $C(0,1)$

To evaluate $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

$M = x^2 - \cosh y \qquad N = y + \sin x$

$\frac{\partial M}{\partial y} = -\sinh y \qquad \frac{\partial N}{\partial x} = \cos x$

$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = \cos x + \sinh y$



Draw a vertical strip PQ in the region

We have to fix x first.

In the region x varies from 0 to π.

$\therefore$ x limits are $x = 0$, $x = \pi$.

For each x, y varies from a point P on x-axis $(y=0)$ to a point Q on the line $y = 1$.

$\therefore$ y limits are $y = 0$, $y = 1$.

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \iint_R (\cos x + \sinh y)\,dx\,dy$$

$$= \int_{x=0}^{x=\pi} \int_{y=0}^{y=1} (\cos x + \sinh y)\,dy\,dx$$

$$= \int_{x=0}^{x=\pi} \left[\cos x \cdot y + \cosh y\right]_{y=0}^{y=1} dx.$$

$$\iint\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dx\,dy = \int_{x=0}^{x=\pi}(\cos x+\cosh 1-1)\,dx$$

$$= \Big[\sin x + x\cosh 1 - x\Big]_{x=0}^{x=\pi}$$

$$\iint\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dx\,dy = \pi(\cosh 1-1) \quad \text{--- (3)}$$

<u>To evaluate $\oint_C M\,dx+N\,dy$</u> :-

The region is rectangle OABC

To evaluate the line integral $\oint_C M\,dx+N\,dy$ ——— (4)

We can write $\oint_C M\,dx+N\,dy = \int_{OA} M\,dx+N\,dy + \int_{AB} M\,dx+N\,dy + \int_{BC} M\,dx+N\,dy + \int_{CO} M\,dx+N\,dy$

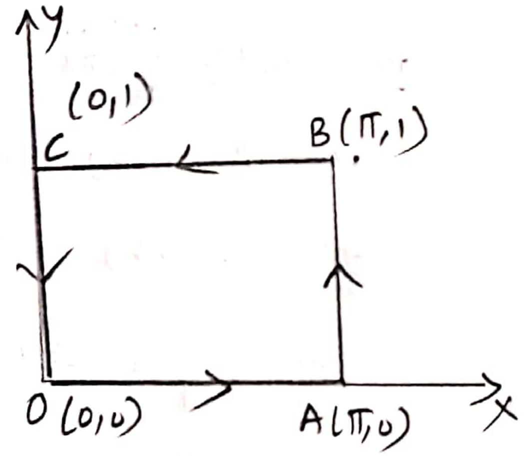
<u>Case (i)</u> <u>To evaluate $\int_{OA} M\,dx+N\,dy$ (OR) Along the line OA</u> :-

We have $O(0,0)$ $A(\pi,0)$

Here $y=0 \Rightarrow dy=0$

x varies from 0 to π.

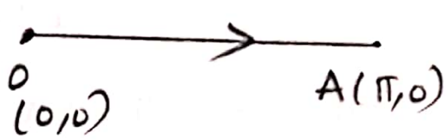
$\therefore$ x limits $x=0,\ x=\pi$.



$$M\,dx+N\,dy = (x^2-\cosh y)\,dx + (y+\sin x)\,dy$$
$$= (x^2-\cosh 0)\,dx \quad [\because y=0,\ dy=0]$$

$$\int_{OA} M\,dx+N\,dy = \int_{OA}(x^2-\cosh 0)\,dx$$

$$= \int_{x=0}^{x=\pi}(x^2-\cosh 0)\,dx$$



$$= \left[\frac{x^3}{3} - x\cosh 0\right]_{x=0}^{x=\pi}$$

$$\int_{OA} M\,dx+N\,dy = \frac{\pi^3}{3} - \pi \quad \text{——— (5)} \quad [\because \cosh 0 = 1]$$

<u>Case (ii)</u> <u>To evaluate $\int_{AB} M\,dx+N\,dy$ (OR) Along the line AB</u> :-

We have $A(\pi,0)$ $B(\pi,1)$

Here $x=\pi \Rightarrow dx=0$

y varies from 0 to 1.

$\therefore$ y limits $y=0, y=1$.

$$M\,dx + N\,dy = y\,dy \quad [\because x=\pi,\ dx=0]$$

$$\int_{AB} M\,dx + N\,dy = \int_{AB} y\,dy$$

$$= \int_{y=0}^{y=1} y\,dy$$

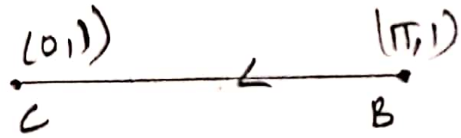
$$= \left[\frac{y^2}{2}\right]_{y=0}^{y=1}$$

$$\int_{AB} M\,dx + N\,dy = \frac{1}{2} \quad \text{—— (6)}$$

<u>Case (iii) To evaluate $\int_{BC} M\,dx + N\,dy$ (OR) Along the line BC</u> :-

We have B$(\pi,1)$ C$(0,1)$

Here $y=1 \Rightarrow dy=0$



x varies from π to 0

$\therefore$ x limits $x=\pi, x=0$

$$M\,dx + N\,dy = (x^2 - \cosh y)\,dx + (y+\sin x)\,dy$$

$$M\,dx + N\,dy = (x^2 - \cosh 1)\,dx \quad [\because y=1,\ dy=0]$$

$$\int_{BC} M\,dx + N\,dy = \int_{BC} (x^2 - \cosh 1)\,dx$$

$$= \int_{x=\pi}^{x=0} (x^2 - \cosh 1)\,dx$$

$$= \left[\frac{x^3}{3} - x\cosh 1\right]_{x=\pi}^{x=0}$$

$$\int_{BC} M\,dx + N\,dy = \pi\cosh 1 - \frac{\pi^3}{3} \quad \text{—— (7)}$$

<u>Case (iv) To evaluate $\int_{CO} M\,dx + N\,dy$ (OR) Along the line CO</u> :-

We have C$(0,1)$ O$(0,0)$

Here $x=0 \Rightarrow dx=0$

y varies from 1 to 0

$\therefore$ y limits $y=1, y=0$.

$$Mdx + Ndy = (x^2 - \cosh y)\,dx + (y + \sin x)\,dy$$

$$Mdx + Ndy = y\,dy$$

$$\int_{CO} Mdx + Ndy = \int_{CO} y\,dy$$

$$= \int_{y=1}^{y=0} y\,dy$$

$$= \left[\frac{y^2}{2}\right]_{y=1}^{y=0}$$

$$\int_{CO} Mdx + Ndy = -\frac{1}{2} \quad \text{—— (8)}$$

sub (5) (6) (7) and (8) in (4), we get

$$\oint_C Mdx + Ndy = \frac{\pi^3}{3} - \pi\cosh 0 + \frac{1}{2} + \pi\cosh 1 - \frac{\pi^3}{3} \quad [\because \cosh 0 = 1]$$

$$\oint_C Mdx + Ndy = \pi(\cosh 1 - 1)$$

$$\therefore \oint_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$\therefore$ Green's theorem verified.

→ Applying Green's theorem to evaluate $\oint_C (2x^2-y^2)\,dx + (x^2+y^2)\,dy$ where C is the boundary of the area enclosed by the x-axis and upper half of circle $x^2+y^2=a^2$

Sol: Given that $I = \oint_C (2x^2-y^2)\,dx + (x^2+y^2)\,dy$ —— ①

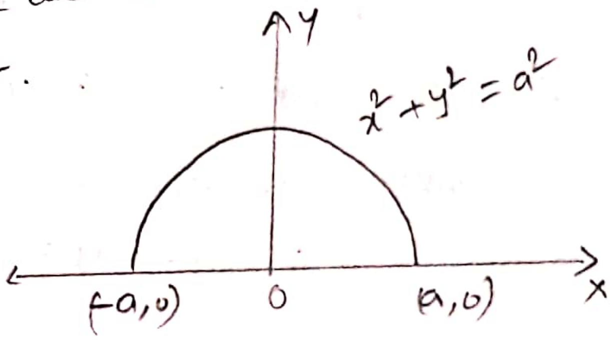
Wkt Green's Theorem in a plane.

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy \quad \text{—— ②}$$

Compare ① with L.H.S of ②, we get

Here $M = 2x^2-y^2$ $\quad N = x^2+y^2$

Given that c is the boundary of the area enclosed by the x-axis and upper half of the circle $x^2+y^2=a^2$.



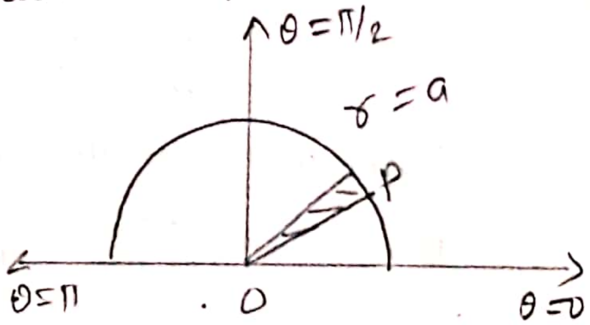
To evaluate $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$:-

$M = 2x^2-y^2$ $\quad N = x^2+y^2$

$\frac{\partial M}{\partial y} = -2y$ $\quad \frac{\partial N}{\partial x} = 2x$

$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 2(x+y)$

To change ~~the order of~~ into polar coordinates put $x = r\cos\theta$, $y = r\sin\theta$ $\quad dx\,dy = r\,dr\,d\theta$.



In the region θ varies from 0 to π

$\therefore$ θ limits $\theta=0$, $\theta=\pi$

Draw a radius vector OP in the region which starts at O $(r=0)$ and terminates at P (which is on the circle $r=a$)

$\therefore$ r limits $r=0$, $r=a$.

$\left[\because x^2+y^2=a^2 \quad r^2(\cos^2\theta+\sin^2\theta)=a^2 \quad r=a.\right]$

$$\oint_C (2x^2-y^2)\,dx + (x^2+y^2)\,dx = \iint_R 2(x+y)\,dx\,dy$$

$$= 2\int_{\theta=0}^{\theta=\pi} \int_{r=0}^{r=a} r(\cos\theta+\sin\theta)\, r\,dr\,d\theta$$

$\left[\because x = r\cos\theta,\ y = r\sin\theta \quad dx\,dy = r\,dr\,d\theta\right.$

$$= 2\int_{r=0}^{r=a} r^2\,dr \int_{\theta=0}^{\theta=\pi} (\cos\theta+\sin\theta)\,d\theta$$

$$= 2\left[\frac{r^3}{3}\right]_{r=0}^{r=a}\left[\sin\theta-\cos\theta\right]_{\theta=0}^{\theta=\pi}$$

$$= \frac{2a^3}{3}\left[(\sin\pi-\cos\pi)-(\sin 0-\cos 0)\right]$$

$$\oint_C (2x^2-y^2)\,dx+(x^2+y^2)\,dy = \frac{4a^3}{3}$$

→ Applying Green's theorem to evaluate $\int_C (x^2+xy)\,dx + (x^2+y^2)\,dy$ where c is the square formed by the lines $x=\pm1$ and $y\pm1$.

sol: Given that $I = \int_C (x^2+xy)\,dx+(x^2+y^2)\,dy$ —— ①

WKt Green's theorem.

$$\oint_C M\,dx+N\,dy = \iint_R \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dx\,dy \quad \text{—— ②}$$

Compare ① with L.H.S of ②,

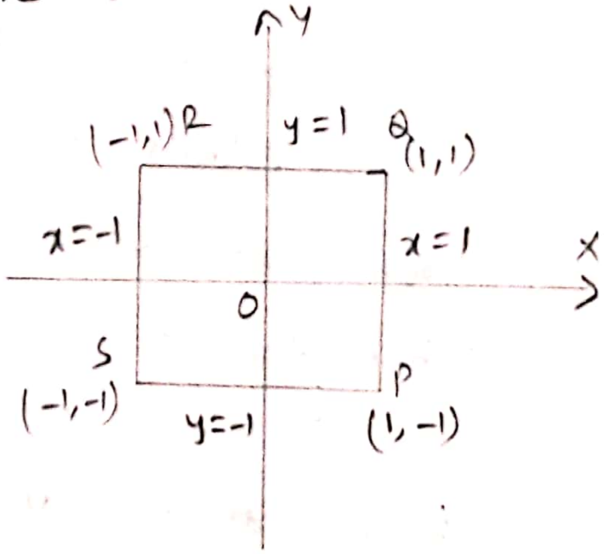
Here $M = x^2+xy$ $\quad N = x^2+y^2$

$\frac{\partial M}{\partial y} = x \quad \frac{\partial N}{\partial x} = 2x$

$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = x.$

Given that c is the square formed by the lines $x=\pm1$, $y=\pm1$

To evaluate $\iint_R \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dx\,dy$:-



In the region x varies from -1 to 1

∴ x limits are $x=-1$, $x=1$.

y varies from -1 to 1

∴ y limits are $y=-1$, $y=1$

$$\oint_C (x^2+xy)\,dx+(x^2+y^2)\,dy = \iint_R x\,dx\,dy$$

$$\oint_C (x^2+xy)dx + (x^2+y^2)\,dy = \int_{x=-1}^{x=1}\int_{y=-1}^{y=1} x\,dx\,dy$$

$$= \int_{x=-1}^{x=1} x\,dx \int_{y=-1}^{y=1} dy$$

$$= \left[\frac{x^2}{2}\right]_{x=-1}^{x=1} \left[y\right]_{y=-1}^{y=1}$$

$$= \left(\tfrac{1}{2}-\tfrac{1}{2}\right)\left[1-(-1)\right]$$

$$\oint_C (x^2+xy)dx + (x^2+y^2)dy = 0$$

→ Applying Green's theorem to evaluate $\int_C e^x \sin y\, dx + e^x \cos y\, dy$. Where c is the rectangle whose vertices are $(0,0)$ $(1,0)$ $(1,\frac{\pi}{2})$ $(0,\frac{\pi}{2})$.

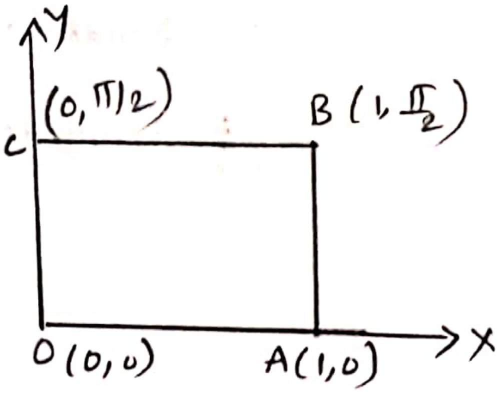
Sol:- Given that $I = \int_C e^x \sin y\, dx + e^x \cos y\, dy$ —— ①

Wkt Greens theorem. $\oint_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$ —— ②

compare ① with L.H.S of ②, we get

$M = e^x \sin y \qquad N = e^x \cos y$

The vertices of rectangle are $O(0,0)$ $A(1,0)$ $B(1,\frac{\pi}{2})$ $C(0,\frac{\pi}{2})$



$$\frac{\partial M}{\partial y} = e^x \cos y \qquad \frac{\partial N}{\partial x} = e^x \cos y$$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0.$$

$$\oint M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$\oint_C (e^x \sin y)dx + (e^x \cos y)dy = \iint_R 0 . dx\,dy$$

$$\oint_C (e^x \sin y\, dx) + (e^x \cos y)dy = 0.$$

→ Use Green's theorem to evaluate $\oint_C x^2(1+y)\,dx + (x^3+y^3)\,dy$ where c is the square bounded by $x=\pm 1$ and $y=\pm 1$.

Sol:- Given that $I = \oint_C x^2(1+y)\,dx + (x^3+y^3)\,dy$ — ①

WKT Green's theorem $\oint_C M\,dx + N\,dy = \iint \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$ — ②

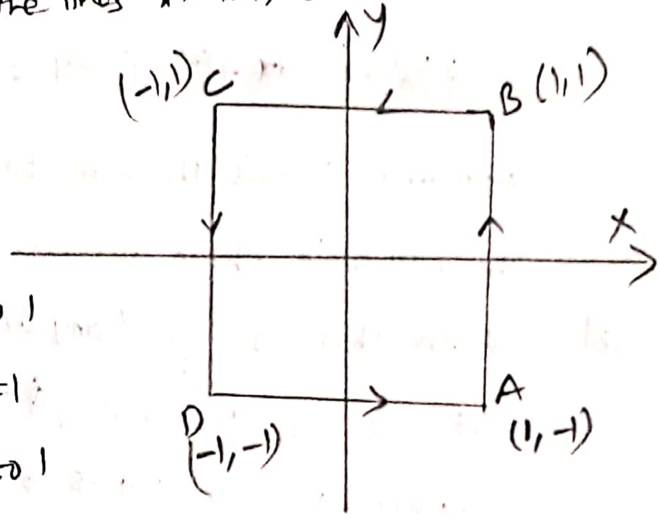
Compare ① with L.H.S ② we get

$M = x^2(1+y) \quad N = x^3+y^3$.

Given that the square bounded by the lines $x=\pm 1$, $y=\pm 1$.

$\frac{\partial M}{\partial y} = x^2 \qquad \frac{\partial N}{\partial x} = 3x^2$

$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 2x^2$



In the region x varies from -1 to 1

$\therefore$ x limits $x=-1,\ x=1$

y varies from -1 to 1

$\therefore$ y limits $y=-1,\ y=1$

$\therefore \oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

$\oint_C x^2(1+y)\,dx + (x^3+y^3)\,dy = \iint_R 2x^2\,dx\,dy$

$= \int_{x=-1}^{x=1}\int_{y=-1}^{y=1} 2x^2\,dx\,dy$

$= 2\int_{x=-1}^{x=1} x^2\,dx \int_{y=-1}^{y=1} dy$

$= 2\left[\frac{x^3}{3}\right]_{x=-1}^{x=1} \left[y\right]_{y=-1}^{y=1}$

$= \frac{2}{3}\left[1-(-1)\right]\left[1-(-1)\right]$

$\oint_C x^2(1+y)\,dx + (x^3+y^3)\,dy = \frac{8}{3}$

Note :- Area of the plane region R bounded by a simple closed curve c.

Wkt Green's theorem.

$$\oint_c M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy$$

Let $M = -y$ $N = x$.

$$\oint_c x dy - y dx = \iint_R (1+1)\, dx\, dy$$

$$\oint_c x dy - y dx = 2 \iint_R dx\, dy.$$

$$\oint_c x dy - y dx = 2\,(\text{Area of the region})$$

$$\therefore \text{Area} = \frac{1}{2} \oint_c x dy - y dx.$$

→ Find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ using Green's theorem.

sol:- Given that $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Parametric equations of ellipse are $x = a\cos\theta$ $y = b\sin\theta$.

$dx = -a\sin\theta\, d\theta$ $dy = b\cos\theta\, d\theta$.

$$\text{Area } A = \frac{1}{2} \oint_c x dy - y dx.$$

θ varies from 0 to 2π

$\therefore$ θ limits $\theta = 0$, $\theta = 2\pi$

$$A = \frac{1}{2} \int_{\theta=0}^{\theta=2\pi} [a\cos\theta(b\cos\theta) - b\sin\theta(-a\sin\theta)]\, d\theta$$

$$= \frac{1}{2} ab \int_{\theta=0}^{\theta=2\pi} (\cos^2\theta + \sin^2\theta)\, d\theta$$

$$= \frac{1}{2} ab \int_{\theta=0}^{\theta=2\pi} d\theta.$$

$$= \frac{1}{2} ab. [\theta]_{\theta=0}^{\theta=2\pi}$$

$$= \frac{1}{2} ab(2\pi - 0)$$

$$\text{Area} = \pi ab.$$

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→ Find the area of the circle $x^2+y^2=r^2$ using Green's Theorem.

Sol: Given that $x^2+y^2=r^2$.

Parametric equations of circle are $x=r\cos\theta$ $y=r\sin\theta$.

$dx=-r\sin\theta\, d\theta$, $dy=r\cos\theta\, d\theta$.

θ varies from 0 to 2π.

$\therefore$ θ limits $\theta=0$, $\theta=2\pi$.

Area $A=\frac{1}{2}\oint_C x\,dy-y\,dx$.

$$A=\frac{1}{2}\int_{\theta=0}^{\theta=2\pi}[r\cos\theta.r\cos\theta-r\sin\theta(-r\sin\theta)]\,d\theta$$

$$=\frac{1}{2}\int_{\theta=0}^{\theta=2\pi}r^2[\cos^2\theta+\sin^2\theta]\,d\theta$$

$$=\frac{r^2}{2}\int_{\theta=0}^{\theta=2\pi}d\theta$$

$$=\frac{r^2}{2}[\theta]_{\theta=0}^{\theta=2\pi}$$

$$=\frac{r^2}{2}[2\pi-0]$$

Area $A=\pi r^2$

Using Green's theorem, evaluate $\int_C xy^2\,dy - x^2y\,dx$. where C is the cardioid $r = a(1-\cos\theta)$.

Sol:- We know that the Green's theorem.

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$\int_C xy^2\,dy - x^2y\,dx = \iint_R \left\{\frac{\partial}{\partial x}(xy^2) - \frac{\partial}{\partial y}(-x^2y)\right\} dx\,dy$$

$$= \iint_R (x^2+y^2)\,dx\,dy$$ Where R is bounded by the cardioid $r = a(1-\cos\theta)$

To change it into polar co ordinates put $x = r\cos\theta$
$y = r\sin\theta$. $dx\,dy = r\,dr\,d\theta$.

In the region R, θ varies from $\theta = 0$ to $\theta = 2\pi$
r varies from $r = 0$ to $r = a(1-\cos\theta)$

$$= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=a(1-\cos\theta)} r^2\, r\,dr\,d\theta = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=a(1-\cos\theta)} r^3\,dr\,d\theta.$$

$$= \int_{\theta=0}^{\theta=2\pi} \left[\frac{r^4}{4}\right]_{r=0}^{r=a(1-\cos\theta)} d\theta$$

$$= \frac{a^4}{4}\int_{\theta=0}^{\theta=2\pi} (1-\cos\theta)^4\,d\theta = \frac{a^4}{4}\int_{\theta=0}^{\theta=2\pi} \left(2\sin^2\frac{\theta}{2}\right)^4 d\theta$$

$$= 4a^4 \int_{\theta=0}^{\theta=2\pi} \sin^8\left(\frac{\theta}{2}\right) d\theta$$

Put $\frac{\theta}{2} = t$
$d\theta = 2dt$
when $\theta = 0, t = 0$
when $\theta = 2\pi, t = \pi$

$$= 4a^4 \int_{t=0}^{t=\pi} \sin^8(t)\,2dt$$

$$= 16a^4 \int_{t=0}^{t=\pi} \sin^8(t)\,dt = 16a^4 \cdot \frac{7}{8}\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}$$

$$= \frac{35}{16}\cdot \pi a^4.$$

Surface Integrals :-

A surface $\bar{r} = \bar{f}(u,v)$ is called a smooth surface if $\bar{f}(u,v)$ is continuous and possesses partial derivatives. Let $\bar{F}(\bar{r})$ be a continuous vector point function, defined over the smooth surface $\bar{r} = \bar{f}(u,v)$. Let S be the region of the surface. Divide the region into m subregions of areas $\delta S_1, \delta S_2, \delta S_3 \ldots \delta S_m$. Let P_i be a point of δS_i and $\bar{n}$ be the unit normal to δS_i at P_i. Let δA_i be the vector area of δS_i. Then $\delta \bar{A}_i = \bar{n}_i \, \delta S_i$

Form the sum $I_m = \sum_{i=1}^{m} \bar{F}(\bar{r}_i)\, \delta A_i = \sum \bar{F}(\bar{r}_i) \cdot \bar{n}_i \, ds_i$

Let m tend to infinity in such a way that each δS_i shrinks to a point. The limit of I_m if it exists is called the normal surface integral of $\bar{F}(\bar{r})$ over the region S of the surface $\bar{r} = \bar{f}(u,v)$ and is denoted by

$\int_S \bar{F}(\bar{r})\, d\bar{A}$ (or) $\int_S \bar{F} \cdot \bar{n}\, ds$.

Note:- other types surface integrals are $\int \bar{F} \times d\bar{A}$ (or) $\int \phi \, d\bar{A}$

Any integral which is to be evaluated over a surface is called a surface integral.

Surface Integrals - Cartesian Form :-

Let $\bar{F}(\bar{r}) = F_1 i + F_2 j + F_3 k$ where F_1, F_2, F_3 are continuous and differentiable functions of x, y, z.

Then $\int_S \bar{F} \cdot \bar{n}\, ds = \iint_S F_1\, dy\, dz + F_2\, dz\, dx + F_3\, dx\, dy$.

Note:- Let R_1 be the projection of S on xy-plane Then

$$\int_S \bar{F} \cdot \bar{n}\, ds = \iint_{R_1} \bar{F} \cdot \bar{n} \frac{dx\, dy}{|\bar{n} \cdot \bar{k}|}$$

Let R_2 be the projection of S on yz-plane Then.

$$\int_S \bar{F} \cdot \bar{n}\, ds = \iint_{R_2} \bar{F} \cdot \bar{n} \frac{dy\, dz}{|\bar{n} \cdot \bar{i}|}$$

Let R_3 be the projection of S on zx-plane Then

$$\int_S \bar{F} \cdot \bar{n}\, ds = \iint_{R_3} \bar{F} \cdot \bar{n} \frac{dx\, dz}{|n \cdot j|}$$

→ Evaluate $\int_S \bar{F}.\bar{n}\, ds$. Where $\bar{F} = 18z\bar{i} - 12\bar{j} + 3y\bar{k}$ and S is the ~~surface of~~ part of the surface of the plane $2x+3y+6z=12$ located in the first octant.

Sol:- Given that $\bar{F} = 18z\bar{i} - 12\bar{j} + 3y\bar{k}$.

We have to find $\int_S \bar{F}.\bar{n}\, ds$

Given that the plane $\phi = 2x+3y+6z-12$.

Normal to the plane ϕ is $\nabla\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$.

$\nabla\phi = 2\bar{i} + 3\bar{j} + 6\bar{k}$.

Unit normal to the surface ϕ is [redacted] $\bar{n} = \frac{\nabla\phi}{|\nabla\phi|}$

$$\bar{n} = \frac{2\bar{i}+3\bar{j}+6\bar{k}}{\sqrt{2^2+3^2+6^2}} = \frac{2\bar{i}+3\bar{j}+6\bar{k}}{7}.$$

Let R be the projection of S on xy-plane Then

$$\int_S \bar{F}.\bar{n}\, ds = \iint_R \bar{F}.\bar{n}\, \frac{dx\,dy}{|\bar{n}.\bar{k}|}$$

Given $\bar{F} = 18z\bar{i} - 12\bar{j} + 3y\bar{k}$

$$\bar{F}.\bar{n} = (18z\bar{i} - 12\bar{j} + 3y\bar{k}) \cdot \left(\frac{2\bar{i}+3\bar{j}+6\bar{k}}{7}\right)$$

$$\bar{F}.\bar{n} = \frac{36z - 36 + 18y}{7} = \frac{6}{7}(6z - 6 + 3y)$$

$$\bar{n}.\bar{k} = \frac{1}{7}(2\bar{i}+3\bar{j}+6\bar{k}).\bar{k} = \frac{6}{7}$$

Given surface is $2x+3y+6z=12$.

∴ Projection R of the plane $\phi = 2x+3y+6z-12=0$ in xy-plane is $2x+3y=12$

[∵ In xy-plane $z=0$]

$$\Rightarrow y = \frac{12-2x}{3}.$$

when $y=0 \Rightarrow x=6$.

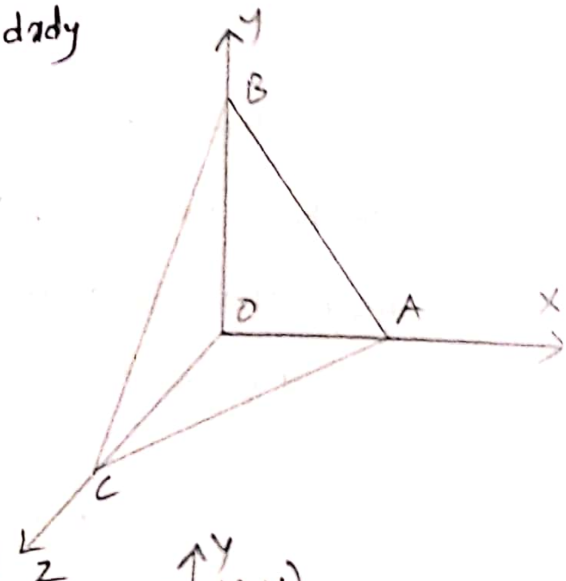
Now x varies from 0 to 6 and y varies from 0 to $\frac{12-2x}{3}$.

$$\int_S \bar{F}.\bar{n}\, ds = \iint_R \bar{F}.\bar{n}.\, \frac{dx\,dy}{|\bar{n}.\bar{k}|}$$

$$= \iint_R \frac{6}{7}(6z-6+3y). \frac{dx\,dy}{\frac{6}{7}}$$

$$\int_S \bar{F}\cdot\bar{n}\, ds = \iint_R (6z - 6 + 3y)\, dx\, dy$$

$$= \iint_R (12 - 2x - 3y - 6 + 3y)\, dx\, dy$$

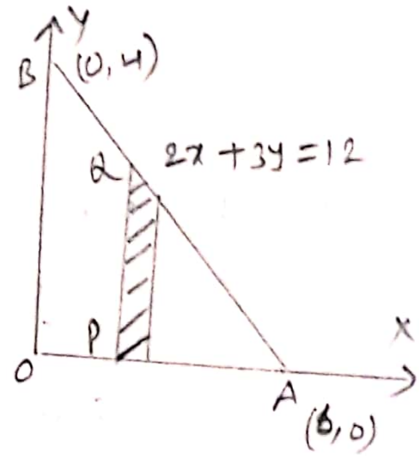


$$= \iint_R (6 - 2x)\, dx\, dy$$

$$= 2\iint_R (3 - 2x)\, dx\, dy$$

$$= 2\int_{x=0}^{x=6} \int_{y=0}^{y=\frac{12-2x}{3}} (3 - x)\, dy \,.\, dx.$$

$$= 2\int_{x=0}^{x=6} (3 - x) \left[y\right]_{y=0}^{y=\frac{12-2x}{3}} dx.$$



$$= 2\int_{x=0}^{x=6} (3 - x) \cdot \frac{1}{3}(12 - 2x)\, dx.$$

$$= \frac{4}{3}\int_{x=0}^{x=6} (3 - x)(6 - x)\, dx.$$

$$= \frac{4}{3}\int_{x=0}^{x=6} (18 - 9x + x^2)\, dx = \frac{4}{3}\left[18x - \frac{9}{2}x^2 + \frac{x^3}{3}\right]_{x=0}^{x=6}$$

$$= \frac{4}{3}\left[18.(6) - \frac{9}{2}(36) + \frac{6^3}{3}\right]$$

$$\int_S \bar{F}\cdot\bar{n}\, ds = 24$$

→ Evaluate $\iint_S \bar{F}.\bar{n}\, ds$ where $\bar{F} = 12x^2y\,i - 3yz\,j + 2z\,k$ and S is the portion of the plane $x+y+z=1$ included in the first octant.

Ans:- $\frac{-55}{24}$

sol:- Given that $\bar{F} = 12x^2y\,i - 3yz\,j + 2z\,k$.

We have to find $\int_S \bar{F}.\bar{n}\, ds$

Given that the plane $\phi = x+y+z-1 = 0$.

Normal to the plane ϕ is $\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$.

$$\nabla\phi = i+j+k.$$

Unit normal to the surface ϕ is $\bar{n} = \frac{\nabla\phi}{|\nabla\phi|}$

$$\bar{n} = \frac{i+j+k}{\sqrt{1^2+1^2+1^2}} = \frac{i+j+k}{\sqrt{3}}$$

Let R be the projection of S on xy-plane Then.

$$\int_S \bar{F}.\bar{n}\, ds = \iint_R \bar{F}.\bar{n}\frac{dx\,dy}{|\bar{n}.\bar{k}|}$$

Given $\bar{F} = 12x^2y\,i - 3yz\,j + 2z\,k$.

$$\bar{F}.\bar{n} = (12x^2y\,i - 3yz\,j + 2z\,k)\cdot\frac{i+j+k}{\sqrt{3}}$$

$$= \frac{12x^2y - 3yz + 2z}{\sqrt{3}}$$

$$\bar{F}.\bar{n} = \frac{1}{\sqrt{3}}\left[12x^2y - 3y(1-x-y) + 2(1-x-y)\right]$$

$$\bar{F}.\bar{n} = \frac{1}{\sqrt{3}}\left[12x^2y + 3xy + 3y^2 - 2x - 5y + 2\right]$$

$$\bar{n}.\bar{k} = \left(\frac{i+j+k}{\sqrt{3}}\right).k = \frac{1}{\sqrt{3}}$$

Given that plane $x+y+z-1=0$.

∴ Projection R of the plane $\phi = x+y+z-1=0$ is in xy-plane is $x+y=1$.

$$\Rightarrow y = 1-x.$$

when $y=0$, $x=1$.

Now x varies from 0 to 1 and y varies from 0 to $1-x$.

$$\int_S F\cdot\bar{n}\,ds = \iint_R F\cdot\bar{n}\,\frac{dx\,dy}{|\bar{n}\cdot\bar{k}|}$$

$$= \iint_R \frac{(12x^2y+3xy+3y^2-2x-5y+2)}{\sqrt{3}}\cdot\frac{dx\,dy}{\frac{1}{\sqrt{3}}}$$

$$= \int_{x=0}^{x=1}\int_{y=0}^{y=1-x}\left[12x^2y+3xy+3y^2-2x-5y+2\right]dy\,dx$$

$$= \int_{x=0}^{x=1}\left[12x^2\cdot\frac{y^2}{2}+3x\cdot\frac{y^2}{2}+3\cdot\frac{y^3}{3}-2xy-5\frac{y^2}{2}+2y\right]_{y=0}^{y=1-x}dx$$

$$= \int_{x=0}^{x=1}\left[\left(6x^2+\frac{3}{2}x-\frac{5}{2}\right)y^2+y^3-2xy+2y\right]_{y=0}^{y=1-x}dx.$$

$$= \int_{x=0}^{x=1}\left[\left(6x^2+\frac{3}{2}x-\frac{5}{2}\right)(1-x)^2+(1-x)^3-2x(1-x)+2(1-x)\right]dx$$

$$= \frac{1}{2}\int_{x=0}^{x=1}(x^3+11x^2-x-8)\,dx$$

$$= \frac{1}{2}\left[\frac{x^4}{4}+11\frac{x^3}{3}-\frac{x^2}{2}-8x\right]_{x=0}^{x=1}$$

$$= \frac{1}{2}\left[\left(\frac{1}{4}+\frac{11}{3}-\frac{1}{2}-8\right)-0\right]$$

$$\int_S F\cdot\bar{n}\,ds = \frac{-55}{24}$$

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→ Evaluate $\int \bar{F}.\bar{n}\, ds$ where $\bar{F} = zi + xj - 3y^2zk$ and S is the surface $x^2+y^2=16$ included in the first octant between $z=0$ and $z=5$.

Sol:- Given that $\bar{F} = zi + xj - 3y^2zk$.

We have to find $\int \bar{F}.\bar{n}\, ds$.

Given that the surface S is $\phi = x^2+y^2-16$

Normal to the surface ϕ is $\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$.

$$\nabla\phi = 2xi + 2yj$$

Unit normal to the surface ϕ is $\bar{n} = \frac{\nabla\phi}{|\nabla\phi|}$

$$\bar{n} = \frac{2(xi+yj)}{2\sqrt{x^2+y^2}} = \frac{xi+yj}{\sqrt{16}}$$

$$\bar{n} = \frac{xi+yj}{4}.$$

Let R be the projection of S on yz plane. Then R is the rectangle OBCD.

$$\iint_S \bar{F}.\bar{n}\, ds = \iint_R \bar{F}.\bar{n}\, \frac{dy\,dz}{|n.i|}$$

$$\bar{F}.\bar{n} = (zi+xj-3y^2zk)\cdot\frac{(xi+yj)}{4} = \frac{xz+xy}{4}$$

$$\bar{n}.i = \frac{1}{4}(xi+yj).i = \frac{x}{4}.$$

For the surface $x^2+y^2=16$ in the yz plane, $x=0 \Rightarrow y=4$.

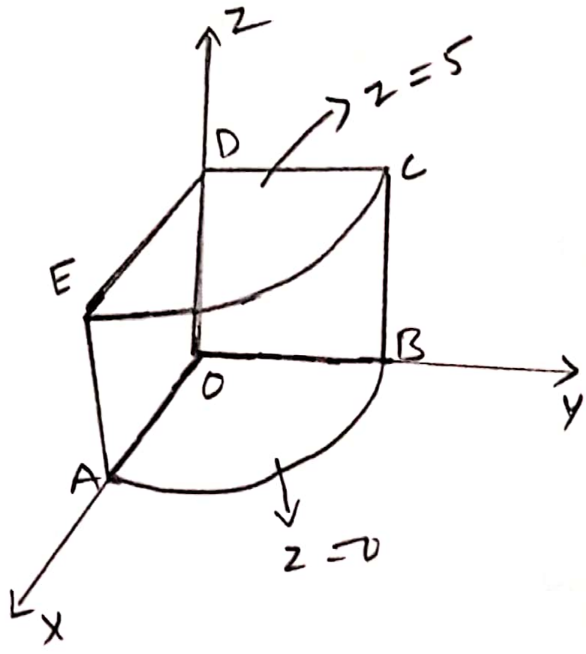
Hence in the first octant y varies from 0 to 4, z varies from 0 to 5.

Then $\iint_S \bar{F}.\bar{n}\, ds = \iint_R \bar{F}.\bar{n}\, \frac{dy\,dz}{|n.i|}$

$$= \iint_R \frac{x(y+z)}{4} \cdot \frac{dy\,dz}{\frac{x}{4}}$$

$$= \int_{y=0}^{y=4}\int_{z=0}^{z=5} (y+z)\, dz.dy$$

$$= \int_{y=0}^{y=4} \left[yz + \frac{z^2}{2}\right]_{z=0}^{z=5} dy$$



$$\iint_S \bar{F}\cdot\bar{n}\,ds = \int_{y=0}^{y=4} \left(5y + \frac{25}{2}\right) dy$$

$$= \left[\frac{5}{2}y^2 + \frac{25}{2}y\right]_{y=0}^{y=4}$$

$$= \left(\frac{5}{2}\cdot 16 + \frac{25}{2}\cdot 4\right) - 0$$

$$= (40 + 50)$$

$$\iint_S \bar{F}\cdot\bar{n}\,ds = 90.$$

→ Evaluate $\iint_S \bar{F}\cdot\bar{n}\,ds$ if $\bar{F} = yz\,i + 2y^2 j + xz^2 k$ and s is the surface of the cylinder $x^2+y^2=9$ contained in the first octant between the planes $z=0$ and $z=2$

Ans:- 78

Sol:- Given that $\bar{F} = yz\,i + 2y^2 j + xz^2 k$.

We have to find $\int_S \bar{F}\cdot\bar{n}\,ds$

Given that the surface S is $\phi = x^2+y^2-9$

Normal to the surface ϕ is $\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$

$$\nabla\phi = 2x\,i + 2y\,j$$

Unit normal to the surface ϕ is $\bar{n} = \frac{\nabla\phi}{|\nabla\phi|}$

$$\bar{n} = \frac{2(xi+yj)}{2\sqrt{x^2+y^2}} = \frac{xi+yj}{3}$$

Let R be the projection of S on YZ plane Then R is the rectangle $OBCD$.

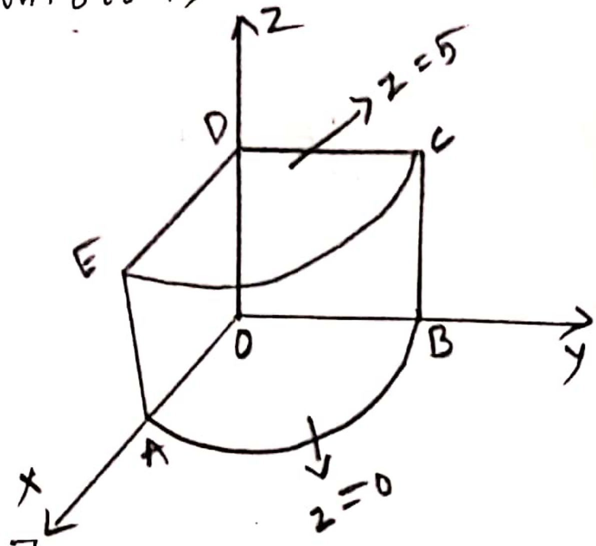
$$\iint_S \bar{F}\cdot\bar{n}\,ds = \iint_R \bar{F}\cdot\bar{n}\,\frac{dy\,dz}{|n\cdot i|}$$

$$\bar{F}\cdot\bar{n} = (yz\,i + 2y^2 j + xz^2 k)\cdot\left(\frac{xi+yj}{3}\right) = \frac{xyz+2y^3}{3}$$

$$n\cdot i = \left(\frac{xi+yj}{3}\right)\cdot i = \frac{x}{3}.$$

For the surface $x^2+y^2=9$ in the YZ plane $x=0 \Rightarrow y=3$.

Hence in the first octanct y varies from 0 to 4, z varies from 0 to 5



$$\text{Then } \iint_S \bar{F}\cdot\bar{n}\,ds = \iint_R \bar{F}\cdot\bar{n}\,\frac{dy\,dz}{|n\cdot i|}$$

$$= \iint_R \frac{xyz+2y^3}{3}\cdot\frac{dy\,dz}{\frac{x}{3}}$$

$$= \iint_R \left(yz + \frac{2y^3}{x}\right)dy\,dz$$

$$= \int_{y=0}^{y=3}\int_{z=0}^{z=5}\left[yz + \frac{2y^3}{\sqrt{9-y^2}}\right]dy\,dz$$

$$\iint_S \bar{F}\cdot\bar{n}\,ds = \int_{y=0}^{y=3}\left[y\cdot\frac{z^2}{2} + \frac{2y^3}{\sqrt{9-y^2}}\,z\right]_{z=0}^{z=2} dy$$

$$= \int_{y=0}^{y=3}\left[2y + \frac{4y^3}{\sqrt{9-y^2}}\right] dy$$

$$= \int_{y=0}^{y=3} 2y\,dy + 4\int_{y=0}^{y=3}\frac{y^3}{\sqrt{9-y^2}}\,dy$$

$$= \left[2\cdot\frac{y^2}{2}\right]_{y=0}^{y=3} + 4\int_{\theta=0}^{\theta=\pi/2}\frac{27\sin^3\theta\;3\cos\theta\,d\theta}{\sqrt{9-9\sin^2\theta}}$$

Put $y = 3\sin\theta$

$dy = 3\cos\theta\,d\theta$

when $y=0$, $\theta=0$

when $y=3$ $\theta=\frac{\pi}{2}$

$$= (9-0) + 4\cdot 27\int_{\theta=0}^{\theta=\pi/2}\sin^3\theta\,d\theta$$

$$= 9 + 4\cdot 27\cdot\int_{\theta=0}^{\theta=\pi/2}\frac{3\sin\theta - \sin 3\theta}{4}\,d\theta$$

$$= 9 + 27\int_{\theta=0}^{\theta=\pi/2}(3\sin\theta - \sin 3\theta)\,d\theta$$

$$= 9 + 27\left[\frac{\cos 3\theta}{3} - 3\cos\theta\right]_{\theta=0}^{\theta=\pi/2}$$

$$= 9 + 27\left[0 - \left(\frac{1}{3} - 3\right)\right]$$

$$= 9 + 27\cdot\frac{8}{3}$$

$$\iint_S \bar{F}\cdot\bar{n}\,ds = 80$$

Stoke's Theorem :- (Transformation between Line Integral and Surface Integral)

Let s be a open surface bounded by a closed, non intersecting curve c. If $\bar{F}$ is any differentiable vector point function then $\oint_c \bar{F}.d\bar{r} = \int_s \text{curl}\bar{F}.\bar{n}ds$ where c is traversed in the positive direction and $\bar{n}$ is unit outward drawn normal at any point of the surface.

Deduction of Green's Theorem from stoke's Theorem :-

Let the surface lie on xy plane. Then z-axis will be along the normal $\bar{n}$ i.e $\bar{n} = \bar{k}$

Let $\bar{F} = F_1\bar{i} + F_2\bar{j}$, $\bar{r} = x\bar{i} + y\bar{j}$

$$\therefore \bar{F}.d\bar{r} = F_1 dx + F_2 dy$$

$$\nabla \times \bar{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & 0 \end{vmatrix}$$

$$= \frac{\partial F_2}{\partial z}\bar{i} - \frac{\partial F_1}{\partial z}\bar{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\bar{k}$$

$$\therefore (\nabla \times \bar{F}).\bar{n} = (\nabla \times \bar{F}).\bar{k} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$$

In xy-plane $ds = dxdy$

$$\therefore \oint_c \bar{F}.d\bar{r} = \int (\nabla \times \bar{F}).\bar{n}\, ds$$

$$\int_c F_1 dx + F_2 dy = \iint_s \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_2}{\partial y}\right) dx\, dy$$

This is same as Green's theorem in a plane.

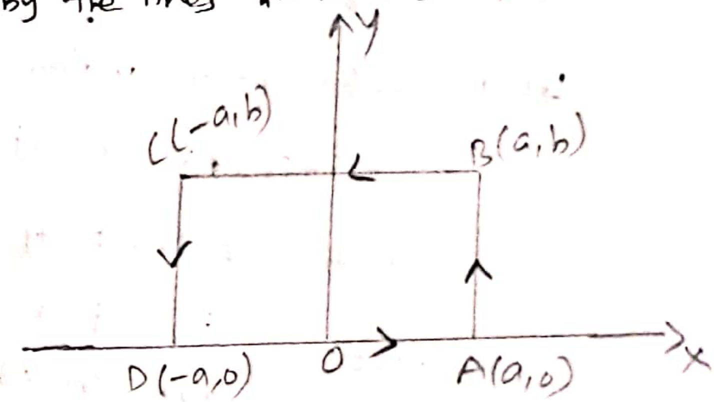
→ Verify Stoke's theorem for $\bar{F} = (x^2+y^2)i - 2xy\,j$ taken round the rectangle bounded by the lines $x = \pm a,\ y=0,\ y=b$.

Sol:- Given that $\bar{F} = (x^2+y^2)i - 2xy\,j$

WKT Stoke's theorem

$$\oint_C \bar{F}.d\bar{r} = \int_S \text{curl}\,\bar{F}.\bar{n}\,ds$$

Given that the rectangle bounded by the lines $x = \pm a,\ y=0,\ y=b$.



To evaluate $\int_S \text{curl}\,\bar{F}.\bar{n}\,ds$:-

$$\bar{F} = (x^2+y^2)i - 2xy\,j$$

We have $\bar{F} = F_1 i + F_2 j + F_3 k$

$F_1 = x^2+y^2 \quad F_2 = -2xy \quad F_3 = 0.$

$$\text{curl}\,\bar{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2+y^2 & -2xy & 0 \end{vmatrix}$$

$$= i\left(\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(-2xy)\right) - j\left(\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}(x^2+y^2)\right) + k\left(\frac{\partial}{\partial x}(-2xy) - \frac{\partial}{\partial y}(x^2+y^2)\right)$$

$$= k(-2y-2y)$$

$$\text{curl}\,\bar{F} = -4y\,k.$$

The rectangle ABCD is in xy-plane. z-axis is perpendicular to xy-plane.

Along z-axis $\bar{k}$ is the unit normal vector.

So $\bar{n} = \bar{k}$

$$\text{curl}\,\bar{F}.\bar{n} = (-4y\,\bar{k}).\bar{k} = -4y.$$

In the region x varies from $-a$ to a

$\therefore$ x limits are $x=-a, x=a$.

y varies from 0 to b

$\therefore$ y limits are $y=0, y=b$

$$\int_S \text{curl}\,\bar{F}\cdot\bar{n}\,ds = \int_S -4y\,dx\,dy$$ [$\because$ Rectangle is in xy-plane. $ds = dx\,dy$]

$$= \int_{x=-a}^{x=a}\int_{y=0}^{y=b} -4y\,dx\,dy$$

$$= \int_{x=-a}^{x=a} dx \int_{y=0}^{y=b} -4y\,dy$$

$$= [x]_{x=a}^{x=a}\left[-4\frac{y^2}{2}\right]_{y=0}^{y=b}$$

$$[a-(-a)]\,[-2b^2-0]$$

$$\int_S \text{curl}\,\bar{F}\cdot\bar{n}\,ds = -4ab^2$$

To evaluate $\oint_C \bar{F}\cdot d\bar{r}$:-

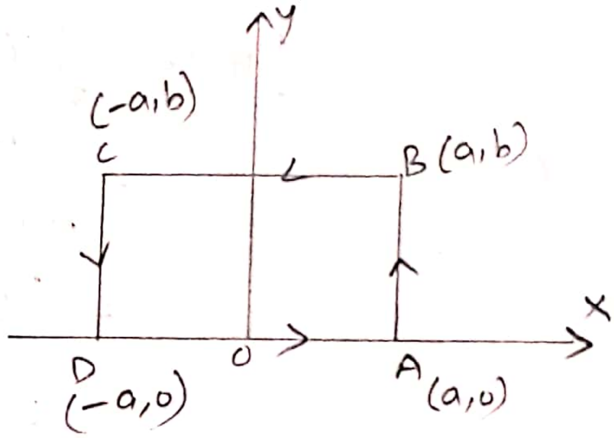
We have $\bar{F} = (x^2+y^2)i - 2xy\,j$

$\bar{r} = xi + yj$ ($\because$ xy-plane)

$d\bar{r} = dx\,i + dy\,j$

$\bar{F}\cdot d\bar{r} = [(x^2+y^2)i - 2xy\,j]\cdot[dx\,i + dy\,j]$

$\bar{F}\cdot d\bar{r} = (x^2+y^2)dx - 2xy\,dy$ —— ①



To evaluate the line integral $\oint_C \bar{F}\cdot d\bar{r}$

We can write $\oint_C \bar{F}\cdot d\bar{r} = \int_{AB} \bar{F}\cdot d\bar{r} + \int_{BC} \bar{F}\cdot d\bar{r} + \int_{CD} \bar{F}\cdot d\bar{r} + \int_{DA} \bar{F}\cdot d\bar{r}$ —— ②

(i) To evaluate $\int_{AB} \bar{F}\cdot d\bar{r}$ (or) Along the line AB :-

We have A (a,0) B (a,b)

$x = a \Rightarrow dx = 0$

y varies from 0 to b.

$\therefore$ y limits $y=0, y=b$

From ① $\bar{F}\cdot d\bar{r} = -2ay\,dy$ [$\because x=a,\ dx=0$]

$$\int_{AB} \bar{F}\cdot d\bar{r} = \int_{AB} -2ay\,dy$$

$$= \int_{y=0}^{y=b} -2ay\,dy$$

$$\int_{AB} \bar{F}\cdot d\bar{r} = \left[-2a\cdot\frac{y^2}{2}\right]_{y=0}^{y=b}$$

$$\int_{AB} \bar{F}\cdot d\bar{r} = -ab^2 \quad \text{--- (3)}$$

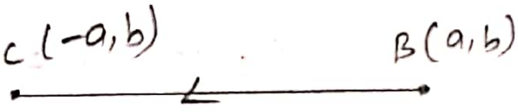
(ii) <u>To evaluate</u> $\int_{BC} \bar{F}\cdot d\bar{r}$ <u>(or) Along the line BC</u> :-

We have $B(a,b)$ $C(-a,b)$

Here $y=b \Rightarrow dy=0$

x varies from $+a$ to $-a$

$\therefore$ x limits are $x=+a$, $x=-a$



From (1) $\bar{F}\cdot d\bar{r} = (x^2+b^2)\,dx$ $\quad [\because y=b,\ dy=0]$

$$\int_{BC} \bar{F}\cdot d\bar{r} = \int_{BC} (x^2+b^2)\,dx$$

$$= \int_{x=+a}^{x=-a} (x^2+b^2)\,dx$$

$$= \left[\frac{x^3}{3}+b^2x\right]_{x=+a}^{x=-a}$$

$$= \left(\frac{-a^3}{3}+ab^2\right) - \left(+\frac{a^3}{3}+ab^2\right)$$

$$\int_{BC} \bar{F}\cdot d\bar{r} = -\frac{2a^3}{3}+2ab^2 \quad \text{--- (4)}$$

(iii) <u>To evaluate</u> $\int_{CD} \bar{F}\cdot d\bar{r}$ <u>(or) Along the line CD</u> :-

We have $C(-a,b)$ $D(-a,0)$

Here $x=-a \Rightarrow dx=0$

y varies from b to 0

$\therefore$ y limits are $y=b$, $y=0$

From (1) $\bar{F}\cdot d\bar{r} = 2ay\,dy$ $\quad [\because x=-a,\ dx=0]$

$$\int_{CD} \bar{F}\cdot d\bar{r} = \int_{CD} 2ay\,dy$$

$$= \int_{y=b}^{y=0} 2ay\,dy$$

$$\int_{CD} \bar{F} \cdot d\bar{r} = \left[2a \frac{y^2}{2} \right]_{y=b}^{y=0}$$

$$= 0 - ab^2$$

$$\int_{CD} \bar{F} \cdot d\bar{r} = -ab^2 \quad \text{—— (5)}$$

(iv) To evaluate $\int_{DA} \bar{F} \cdot d\bar{r}$ (OR) Along the line DA :—

We have $D(-a,0)$ $A(a,0)$.

We have $y = 0 \Rightarrow dy = 0$.

x varies from $-a$ to a.

$\therefore$ x limits are $x = -a, x = a$.

From (1) $\bar{F} \cdot d\bar{r} = x^2 \, dx$ $[\because y = 0, dy = 0]$

$$\int_{DA} \bar{F} \cdot d\bar{r} = \int_{DA} x^2 \, dx$$

$$= \int_{x=-a}^{x=a} x^2 \, dx$$

$$= \left[\frac{x^3}{3} \right]_{x=-a}^{x=a}$$

$$= \frac{a^3}{3} - \frac{(-a)^3}{3}$$

$$\int_{DA} \bar{F} \cdot d\bar{r} = \frac{2a^3}{3} \quad \text{—— (6)}$$

Sub. (3) (4) (5) and (6) in (2), we get

$$\oint_C \bar{F} \cdot d\bar{r} = -ab^2 - \frac{2a^3}{3} - 2ab^2 - ab^2 + \frac{2a^3}{3}$$

$$\oint_C \bar{F} \cdot d\bar{r} = -4ab^2 \quad \text{—— (7)}.$$

$$\therefore \oint_C \bar{F} \cdot d\bar{r} = \int_S \text{curl} \bar{F} \cdot \bar{n} \, ds$$

$\therefore$ stokes theorem verified

→ Verify stokes theorem for $\bar{F} = y^2 i - 2xy j$ taken round the rectangle bounded by $x = \pm b$, $y = 0$, $y = a$.

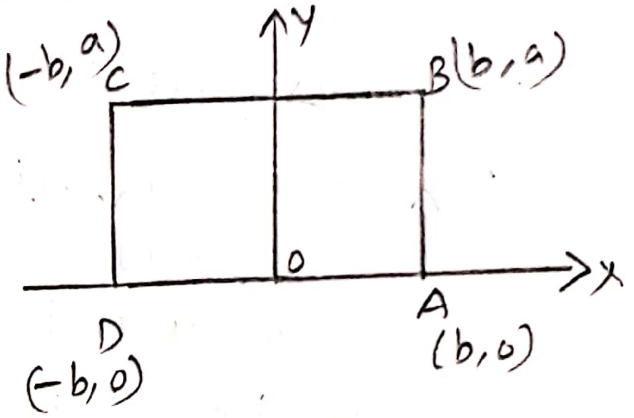
Ans:: $-4a^2b$.

Sol:- Given that $\bar{F} = y^2 i - 2xy j$

Wkt stokes theorem

$$\oint_C \bar{F} \cdot d\bar{r} = \int_S \text{curl}\,\bar{F} \cdot \bar{n}\, ds$$

Given that the rectangle bounded by the lines $x = \pm b$, $y = 0$, $y = a$.



To evaluate $\int_S \text{curl}\,\bar{F} \cdot \bar{n}\, ds$.

$$\bar{F} = y^2 i - 2xy j$$

We have, $\bar{F} = F_1 i + F_2 j + F_3 k$.

$F_1 = y^2$ $F_2 = -2xy$ $F_3 = 0$,

$$\text{Curl}\,\bar{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2 & -2xy & 0 \end{vmatrix}$$

$$\text{Curl}\,\bar{F} = -4y\bar{k}$$

The rectangle ABCD is in XY-plane. z-axis is perpendicular to xy-plane.

Along z-axis $\bar{k}$ is the unit normal vector.

$$\text{So } \bar{n} = \bar{k}$$

$$\text{Curl}\,\bar{F} \cdot \bar{n} = (-4y\bar{k}) \cdot \bar{k} = -4y$$

In the region x varies from $-b$ to b

$\therefore$ x limits are $x = -b, x = b$.

y varies from 0 to a.

$\therefore$ y limits are $y = 0, y = a$.

Rectangle is in XY-plane $ds = dx\,dy$.

$$\int_S \text{curl}\,\bar{F} \cdot \bar{n}\, ds = \int_S -4y\, dx\,dy$$

$$= \int_{x=-b}^{x=b} \int_{y=0}^{y=a} -4y\, dx\,dy$$

$$\int_S \text{curl}\,\bar{F}\cdot\bar{n}\,ds = \int_{x=-b}^{x=b} dx \int_{y=0}^{y=a} -4y\,dy$$

$$= \left[x\right]_{x=-b}^{x=b} \left[-\frac{4y^2}{2}\right]_{y=0}^{y=a}.$$

$$= \left[b-(-b)\right]\left(-2a^2-0\right)$$

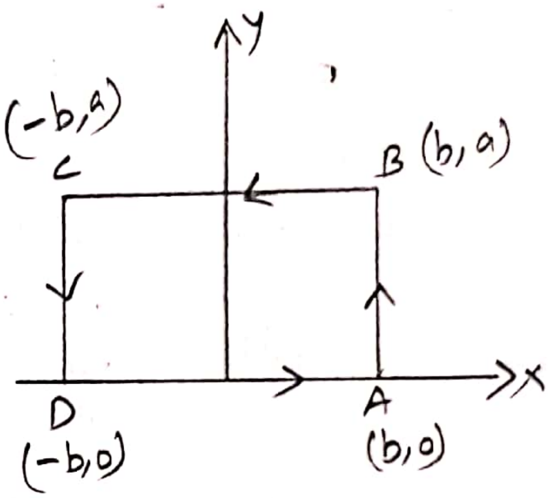
$$\int_S \text{curl}\,\bar{F}\cdot\bar{n}\,ds = -4a^2b$$

<u>To evaluate $\oint_C \bar{F}\cdot d\bar{r}$</u> :-

We have $\bar{F} = y^2 i - 2xy\,j$

$\bar{r} = x i + y j$

$d\bar{r} = dx\, i + dy\, j$



$\bar{F}\cdot d\bar{r} = \left[y^2 i - 2xy\,j\right]\cdot\left[dx\,i + dy\,j\right]$

$\bar{F}\cdot d\bar{r} = y^2 dx - 2xy\,dy$ —— ①.

To evaluate the line integral $\oint_C \bar{F}\cdot d\bar{r}$

We can write $\oint_C \bar{F}\cdot d\bar{r} = \int_{AB}\bar{F}\cdot d\bar{r} + \int_{BC}\bar{F}\cdot d\bar{r} + \int_{CD}\bar{F}\cdot d\bar{r} + \int_{DA}\bar{F}\cdot d\bar{r}$ —— ②.

(i) <u>To evaluate $\int_{AB}\bar{F}\cdot d\bar{r}$ (or) Along the line AB</u> :-

We have A(b, 0) B(b, a)

Here $x = b \Rightarrow dx = 0$.

y varies from 0 to a.

$\therefore$ y limits $y=0,\ y=a$.

From ① $\bar{F}\cdot d\bar{r} = -2by\,dy$.

$$\int_{AB}\bar{F}\cdot d\bar{r} = \int_{AB} -2by\,dy$$

$$= -2b\int_{y=0}^{y=a} y\,dy$$

$$= -2b\left[\frac{y^2}{2}\right]_{y=0}^{y=a}$$

$$\int_{AB}\bar{F}\cdot d\bar{r} = -a^2b \quad \text{—— ③.}$$

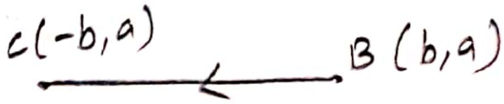
(ii) To evaluate $\int_{BC} \bar{F}\cdot d\bar{r}$ (or) Along the line BC :-

We have $B(b,a)$ $C(-b,a)$

Here $y=a \Rightarrow dy=0$.

x varies from b to $-b$

$\therefore$ x limits are $x=b, x=-b$.



From (1) $\bar{F}\cdot d\bar{r} = a^2\,dx$.

$$\int_{BC} \bar{F}\cdot d\bar{r} = \int_{BC} a^2\,dx$$

$$= a^2 \int_{x=b}^{x=-b} dx$$

$$= a^2 \left[x\right]_{x=b}^{x=-b}$$

$$= a^2(-b-b)$$

$$\int_{BC} \bar{F}\cdot d\bar{r} = -2a^2b \quad \text{——— (4)}$$

(iii) To evaluate $\int_{CD} \bar{F}\cdot d\bar{r}$ (or) Along the line CD :-

We have $C(-b,a)$ $D(-b,0)$

Here $x=-b \Rightarrow dx=0$.

y varies from a to 0.

$\therefore$ y limits are $y=a$, $y=0$.

From (1) $\bar{F}\cdot d\bar{r} = 2by\,dy$

$$\int_{CD} \bar{F}\cdot d\bar{r} = \int_{CD} 2by\,dy$$

$$= 2b \int_{y=a}^{y=0} y\,dy$$

$$= 2b \left[\frac{y^2}{2}\right]_{y=a}^{y=0}$$

$$\int_{CD} \bar{F}\cdot d\bar{r} = -a^2b \quad \text{——— (5)}$$

(iv) To evaluate $\int_{DA} \bar{F}\cdot d\bar{r}$ (or) Along the line DA :-

Here $D(-b,0)$ $A(b,0)$.

$y=0 \Rightarrow dy=0$.

x varies from $-b$ to b.

$\therefore$ x limits $x=-b$, $x=b$.

From (1) $\bar{F}\cdot d\bar{r}=0$

$$\int_{DA} \bar{F}\cdot d\bar{r} = 0. \qquad \text{---} (6)$$

Sub. (3) (4) (5) and (6) in (2), we get

$$\oint_C \bar{F}\cdot d\bar{r} = -a^2b - 2a^2b - a^2b$$

$$\oint_C \bar{F}\cdot d\bar{r} = -4a^2b$$

$$\therefore \oint_C \bar{F}\cdot d\bar{r} = \int_S \text{curl}\,\bar{F}\cdot\bar{n}\,ds$$

$\therefore$ Stokes Theorem verified.

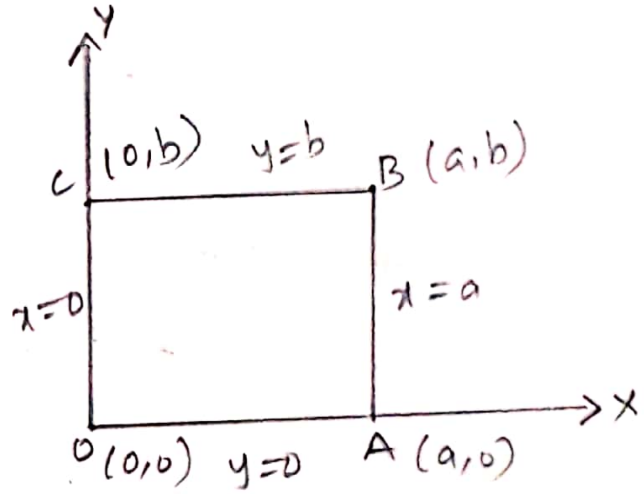
→ Verify Stoke's theorem for $\bar{F} = (x^2-y^2)i + 2xyj$ over the box bounded by the planes $x=0, x=a, y=0, y=b$.

Sol:- Given that $\bar{F} = (x^2-y^2)i + 2xyj$

WKT Stoke's theorem

$$\oint_C \bar{F}\cdot d\bar{r} = \int_S \text{Curl}\,\bar{F}\cdot\bar{n}\,ds.$$

Given that the box bounded by the planes $x=0, x=a, y=0, y=b$.



To evaluate $\int_S \text{Curl}\,\bar{F}\cdot\bar{n}\,ds$:-

$$\bar{F} = (x^2-y^2)i + 2xyj$$

We have $\bar{F} = F_1 i + F_2 j + F_3 k$.

$F_1 = x^2-y^2 \quad F_2 = 2xy \quad F_3 = 0$

$$\text{Curl}\,\bar{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2-y^2 & 2xy & 0 \end{vmatrix}$$

$$= i\left(\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(2xy)\right) - j\left(\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}(x^2-y^2)\right) + k\left(\frac{\partial}{\partial x}(2xy) - \frac{\partial}{\partial y}(x^2-y^2)\right)$$

$$= k(2y+2y)$$

$$\text{Curl}\,\bar{F} = 4yk.$$

The rectangle OABC is in xy-plane. z-axis is perpendicular to xy-plane. Along z-axis, $\bar{k}$ is the unit normal vector.

so $\bar{n} = \bar{k}$

$$\text{Curl}\,\bar{F}\cdot\bar{n} = (4y\bar{k})\cdot\bar{k} = 4y$$

In the region x varies from 0 to a.

$\therefore$ x limits are $x=0, x=a$.

y varies from 0 to b.

$\therefore$ y limits are $y=0, y=b$.

$$\int_S \text{curl}\,\bar{F}\cdot\bar{n}\,ds = \int_S (4y\bar{k})\cdot\bar{k}\,dx\,dy \qquad \left[\because \text{ Rectangle is in } xy\text{-plane } ds = dx\,dy\right]$$

$$= \int_{x=0}^{x=a}\int_{y=0}^{y=b} 4y\,dx\,dy$$

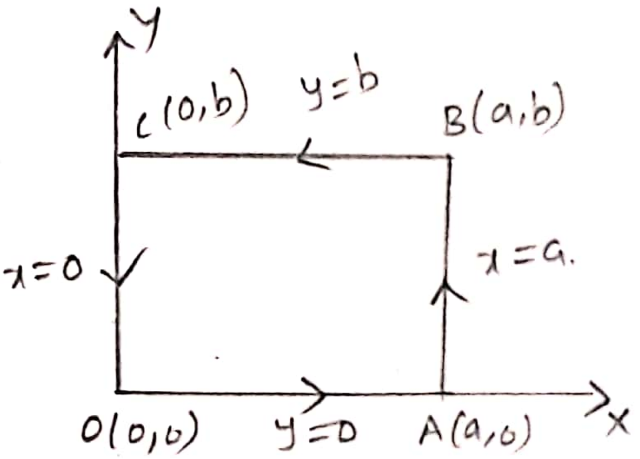
$$= 4\int_{x=0}^{x=a} dx \int_{y=0}^{y=b} y\,dy$$

$$= 4\,[x]_{x=0}^{x=a}\left[\frac{y^2}{2}\right]_{y=0}^{y=b}$$

$$= 4(a-0)\left(\frac{b^2}{2}-0\right)$$

$$\int_S \text{curl}\,\bar{F}\cdot\bar{n}\,ds = 2ab^2$$

To evaluate $\oint_C \bar{F}\cdot d\bar{r}$:-



We have $\bar{F} = (x^2+y^2)i + 2xy\,j$

$\bar{r} = xi + yj$ ($\because$ xy-plane)

$d\bar{r} = dx\,i + dy\,j$

$\bar{F}\cdot d\bar{r} = \left[(x^2-y^2)i + 2xy\,j\right]\cdot\left[dx\,i + dy\,j\right]$

$\bar{F}\cdot d\bar{r} = (x^2-y^2)dx + 2xy\,dy$ ——— (1)

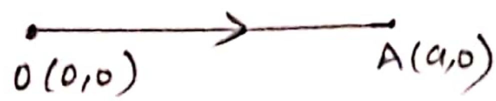
To evaluate the line integral $\oint_C \bar{F}\cdot d\bar{r}$

We can write $\oint_C \bar{F}\cdot d\bar{r} = \int_{OA}\bar{F}\cdot d\bar{r} + \int_{AB}\bar{F}\cdot d\bar{r} + \int_{BC}\bar{F}\cdot d\bar{r} + \int_{CO}\bar{F}\cdot d\bar{r}$ ——— (2).

Case (i) :- To evaluate $\int_{OA}\bar{F}\cdot d\bar{r}$ (or) Along the line OA :-

We have O(0,0) A(a,0)

Here $y=0 \Rightarrow dy=0$.



x varies from 0 to a.

$\therefore$ x limits $x=0,\ x=a$.

From (1) $\bar{F}\cdot d\bar{r} = x^2\,dx$. $[\because y=0,\ dy=0]$

$$\int_{OA}\bar{F}\cdot d\bar{r} = \int_{OA} x^2\,dx$$

$$= \int_{x=0}^{x=a} x^2\,dx.$$

$$\int_{OA} \bar{F}\cdot d\bar{r} = \left[\frac{x^3}{3}\right]_{x=0}^{x=a}$$

$$\int_{OA} \bar{F}\cdot d\bar{r} = \frac{a^3}{3} \quad \text{——} (3).$$

Case(ii) To evaluate $\int_{AB} \bar{F}\cdot d\bar{r}$ (or) Along the line AB :-

We have $A(a,0)$ $B(a,b)$

Here $x=a \Rightarrow dx=0$

y varies from 0 to b.

$\therefore$ y limits $y=0$ $y=b$.

From (1) $\quad \bar{F}\cdot d\bar{r} = 2ay\,dy \quad [\because x=a,\ dx=0]$

$$\int_{AB} \bar{F}\cdot d\bar{r} = \int_{AB} 2ay\,dy$$

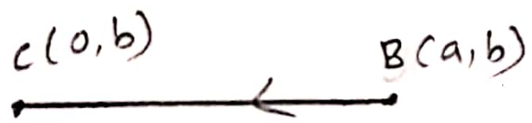
$$= \int_{y=0}^{y=b} 2ay\,dy$$

$$= 2a.\left[\frac{y^2}{2}\right]_{y=0}^{y=b}$$

$$\int_{AB} \bar{F}\cdot d\bar{r} = ab^2 \quad \text{——} (4).$$

Case (iii) To evaluate $\int_{BC} \bar{F}\cdot d\bar{r}$ (or) Along the line BC :-

We have $B(a,b)$ $C(0,b)$.



Here $y=b \Rightarrow dy=0$.

x varies from a to 0.

$\therefore$ x limits $x=a$, $x=0$.

From (1) $\quad \bar{F}\cdot d\bar{r} = (x^2-b^2)\,dx \quad [\because y=b,\ dy=0]$

$$\int_{BC} \bar{F}\cdot d\bar{r} = \int_{BC} (x^2-b^2)\,dx$$

$$= \int_{x=a}^{x=0} (x^2-b^2)\,dx.$$

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$$\int_{BC} \bar{F}\cdot d\bar{r} = \left[\frac{x^3}{3} - b^2 x\right]_{x=a}^{x=0}$$

$$= 0 - \left(\frac{a^3}{3} - ab^2\right)$$

$$\int_{BC} \bar{F}\cdot d\bar{r} = ab^2 - \frac{a^3}{3} \quad \text{— (5)}$$

<u>Case (iv)</u> To evaluate $\int_{CO} \bar{F}\cdot d\bar{r}$ (or) Along the line CO :-

We have $C(0,b)$ $O(0,0)$

Here. $x=0 \Rightarrow dx=0$.

y varies from b to 0.

$\therefore$ y limits $y=b$, $y=0$.

From (1), $\bar{F}\cdot d\bar{r} = 0$ $\quad [\because x=0, dx=0]$.

$$\int_{CO} \bar{F}\cdot d\bar{r} = 0 \quad \text{— (6)}$$

sub. (3), (4) (5) and (6) in (2), we get.

$$\oint_C \bar{F}\cdot d\bar{r} = \frac{a^3}{3} + ab^2 + ab^2 - \frac{a^3}{3} + 0.$$

$$\oint_C \bar{F}\cdot d\bar{r} = 2ab^2.$$

$$\therefore \oint_C \bar{F}\cdot d\bar{r} = \int_S \text{curl}\,\bar{F}\cdot\bar{n}\, ds.$$

$\therefore$ stokes theorem verified.

→ Verify Stoke's theorem for $\bar{F} = (2x-y)i - yz^2 j - y^2 z \bar{k}$ over the upper half surface of the sphere $x^2+y^2+z^2=1$ bounded by the projection of the xy-plane.

Sol:- Given that $\bar{F} = (2x-y)i - yz^2 j - y^2 z k$.

WKT Stoke's theorem $\oint_c \bar{F}.d\bar{r} = \int_s \text{curl}\,\bar{F}.\bar{n}\,ds$.

Given that the upper half surface of the sphere $x^2+y^2+z^2=1$ bounded by the projection of the xy-plane.

To evaluate $\int_s \text{curl}\,\bar{F}.\bar{n}\,ds$:-

$$\bar{F} = (2x-y)i - yz^2 j - y^2 z k.$$

We have $\bar{F} = F_1 i + F_2 j + F_3 k$.

$$F_1 = 2x-y \quad F_2 = -yz^2 \quad F_3 = -y^2 z.$$

$$\text{curl}\,\bar{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2x-y & -yz^2 & -y^2 z \end{vmatrix}$$

$$= i\left(\frac{\partial}{\partial y}(-y^2 z) - \frac{\partial}{\partial z}(-yz^2)\right) - j\left(\frac{\partial}{\partial x}(-y^2 z) - \frac{\partial}{\partial z}(2x-y)\right) + k\left(\frac{\partial}{\partial x}(-yz^2) - \frac{\partial}{\partial y}(2x-y)\right)$$

$$= i(-2yz+2yz) - j(0+0) + k(0+1)$$

$$\text{curl}\,\bar{F} = \bar{k}.$$

The region R is the projection of S on xy-plane.

z-axis is perpendicular xy-plane. so along z-axis unit normal vector $\bar{n} = \bar{k}$

$$\therefore \text{Curl}\,\bar{F}.\bar{n} = \bar{k}.\bar{k} = 1.$$

$$ds = dx\,dy \quad [\because \text{In xy-plane}]$$

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$$\int_S \text{curl}\,\bar{F}\cdot\bar{n}\,ds = \int_S k\cdot k\,dx\,dy = \iint_R dx\,dy.$$

$$= \text{Area of the circle.}$$

$$\int_S \text{curl}\,\bar{F}\cdot\bar{n}\,ds = \pi.$$

$\because\ x^2+y^2+z^2=1$

In xy-plane $z=0$

$x^2+y^2=1$

which is circle and radius $r=1$

To evaluate $\oint_C \bar{F}\cdot d\bar{r}$:-

We have $\bar{F} = (2x-y)i - yz^2 j - y^2 z k$.

In xy plane $z=0$

$$\bar{F} = (2x-y)i$$

$$\bar{r} = xi + yj$$

$$d\bar{r} = dx\,i + dy\,j$$

$$\bar{F}\cdot d\bar{r} = [(2x-y)i + 0j + 0k]\cdot[dx\,i + dy\,j]$$

$$\bar{F}\cdot d\bar{r} = (2x-y)dx$$

The boundary c of s is a circle in xy-plane i.e $x^2+y^2=1$, $z=0$.

The parametric equations of circle are $x=\cos\theta$ $y=\sin\theta$

$dx = -\sin\theta\,d\theta$ $dy = \cos\theta\,d\theta$

θ varies from 0 to 2π

θ limits $\theta=0$, $\theta=2\pi$.

$$\bar{F}\cdot d\bar{r} = (2\cos\theta - \sin\theta)(-\sin\theta)\,d\theta$$

$$\bar{F}\cdot d\bar{r} = (\sin^2\theta - \sin 2\theta)\,d\theta$$

$$\oint_C \bar{F}\cdot d\bar{r} = \int_C (\sin^2\theta - \sin 2\theta)\,d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi}\left[\frac{1-\cos 2\theta}{2} - \sin 2\theta\right]d\theta$$

$$= \frac{1}{2}\int_{\theta=0}^{\theta=2\pi} d\theta - \frac{1}{2}\int_{\theta=0}^{\theta=2\pi}\cos 2\theta\,d\theta - \int_{\theta=0}^{\theta=2\pi}\sin 2\theta\,d\theta$$

$$= \frac{1}{2}[0]_{\theta=0}^{\theta=2\pi} - \frac{1}{2}\left[\frac{\sin 2\theta}{2}\right]_{\theta=0}^{\theta=2\pi} + \left[\frac{\cos 2\theta}{2}\right]_{\theta=0}^{\theta=2\pi}$$

$$\oint_C \bar{F}\cdot d\bar{r} = \frac{1}{2}(2\pi - 0) - \frac{1}{2}\left[\frac{\sin 4\pi}{2} - 0\right] + \left[\frac{\cos 4\pi}{2} - \frac{\cos 0}{2}\right]$$

$$\oint_C \bar{F}\cdot d\bar{r} = \pi .$$

$$\therefore \oint_C \bar{F}\cdot d\bar{r} = \int_S \text{curl}\,\bar{F}\cdot\bar{n}\, ds .$$

$\therefore$ Stoke's theorem verified.

→ Apply stokes theorem, to evaluate $\oint_C (y\,dx + z\,dy + x\,dz)$ where c is the curve of intersection of the sphere $x^2+y^2+z^2=a^2$ and $x+z=a$.

Sol:- Given that $\oint_C (y\,dx + z\,dy + x\,dz)$

Let $\bar{F}\cdot d\bar{r} = y\,dx + z\,dy + x\,dz$

$\bar{F}\cdot d\bar{r} = (yi + zj + xk)\cdot(i\,dx + j\,dy + k\,dz)$

$\Rightarrow \bar{F} = yi + zj + xk \qquad d\bar{r} = i\,dx + j\,dy + k\,dz.$

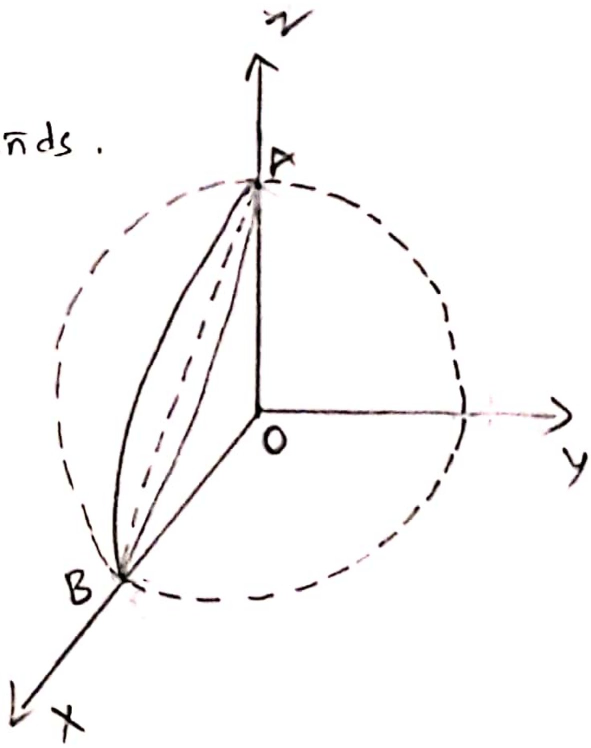
The intersection of the sphere $x^2+y^2+z^2=a^2$ and the plane $x+z=a$ is a circle in the plane $x+z=a$ with AB as diameter

Equation of the plane is $x+z=a \Rightarrow \frac{x}{a}+\frac{z}{a}=1.$

$\therefore OA = OB = a$ i.e $A(a,0,0)$ and $B(0,0,a)$.

$\therefore$ Length of the diameter $AB = \sqrt{a^2+0+a^2} = a\sqrt{2}$

Radius of the circle $r = \frac{a}{\sqrt{2}}$.



W.k.t Stoke's theorem $\oint_C \bar{F}\cdot d\bar{r} = \int_S \text{curl}\,\bar{F}\cdot\bar{n}\,ds$.

$$\text{curl}\,\bar{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix}$$

We have $\bar{F} = F_1 i + F_2 j + F_3 k$

i.e $\bar{F} = yi + zj + xk$

$$\text{curl}\,\bar{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & z & x \end{vmatrix}$$

$\text{Curl}\,\bar{F} = i\left(\frac{\partial}{\partial y}(x) - \frac{\partial}{\partial z}(z)\right) - j\left(\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial z}(y)\right) + k\left(\frac{\partial}{\partial x}(z) - \frac{\partial}{\partial y}(y)\right)$

$\text{Curl}\,F = -i - j - k.$

Let $\bar{n}$ be the unit normal to the surface ϕ. $\bar{n} = \frac{\nabla\phi}{|\nabla\phi|}$

Let $\phi = x + z - a$.

$$\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$

$$\nabla\phi = i + k, \quad |\nabla\phi| = \sqrt{2}.$$

$$\bar{n} = \frac{i+k}{\sqrt{2}}$$

$$\oint_C \bar{F}\cdot d\bar{r} = \int_S \text{Curl}\,\bar{F}\cdot\bar{n}\,ds$$

$$= -\int_S (i+j+k)\cdot\frac{i+k}{\sqrt{2}}\,ds$$

$$= -\frac{1}{\sqrt{2}}\int_S (1+1)\,ds$$

$$= -\sqrt{2}\int_S ds$$

$= -\sqrt{2}$. Area of the circle

$$= -\sqrt{2}\cdot\frac{\pi a^2}{2}$$

$$= \frac{\pi a^2}{\sqrt{2}}.$$

→ Use stoke's theorem to evaluate $\int_S \text{curl}\,\bar{F}.\bar{n}\,ds$ where $\bar{F} = 2y\,i + (x-2zx)j + xy\,k$ and S is the surface of the sphere $x^2+y^2+z^2 = a^2$ above the xy-plane.

Sol: Given that $\bar{F} = 2y\,i + (x-2zx)\,j + xy\,k$

We have $\bar{F} = F_1 i + F_2 j + F_3 k$. $F_1 = 2y$ $F_2 = x-2xz$ $F_3 = xy$.

Given that S is the surface of the sphere $x^2+y^2+z^2 = a^2$ above the xy-plane.

Wkt stokes theorem $\oint_c \bar{F}.d\bar{r} = \int_S \text{curl}\,\bar{F}.\bar{n}\,ds$.

The boundary c of the surface S is the circle $x^2+y^2 = a^2$, $z=0$

[$\because$ In xy-plane $z=0$]

The parametric equations of c are $x = a\cos\theta$ $y = a\sin\theta$, $z=0$

$dx = -a\sin\theta\,d\theta$ $dy = a\cos\theta\,d\theta$

$\bar{F} = 2y\,i + x\,j + xy\,k$ [$\because$ $z=0$ in xy-plane]

$\bar{F} = 2a\sin\theta\,i + a\cos\theta\,j + a^2\sin\theta\cos\theta\,k$.

We have $\bar{r} = x\,i + y\,j$ [$\because$ In xy-plane $z=0$]

$d\bar{r} = dx\,i + dy\,j$

$d\bar{r} = [-a\sin\theta\,i + a\cos\theta\,j]\,d\theta$

$\bar{F}.d\bar{r} = [2a\sin\theta\,i + a\cos\theta\,j + a^2\sin\theta\cos\theta\,k].[-a\sin\theta\,d\theta\,i + a\cos\theta\,d\theta\,j]$

$\bar{F}.d\bar{r} = -2a^2\sin^2\theta\,d\theta + a^2\cos^2\theta\,d\theta$.

$\bar{F}.d\bar{r} = \left[-a^2(1-\cos 2\theta) + \frac{a^2}{2}(1+\cos 2\theta)\right]d\theta$

θ varies from 0 to 2π.

$\therefore$ θ limits $\theta = 0$, $\theta = 2\pi$.

$$\oint_c \bar{F}.d\bar{r} = \int_{\theta=0}^{\theta=2\pi} \left[-a^2(1-\cos 2\theta) + \frac{a^2}{2}(1+\cos 2\theta)\right]d\theta$$

$$= \left[-a^2\left(\theta - \frac{\sin 2\theta}{2}\right) + \frac{a^2}{2}\left(\theta + \frac{\sin 2\theta}{2}\right)\right]_{\theta=0}^{\theta=2\pi}$$

$$= -\pi a^2.$$

→ Use stokes theorem evaluate $\int_c [(x+y)dx + (2x-z)dy + (y+z)dz]$ where c is the boundary of the triangle with vertices $(2,0,0)$ $(0,3,0)$ and $(0,0,6)$.

Sol.- Given that $\int_c (x+y)dx + (2x-z)dy + (y+z)dz$

Let $\bar{F}.d\bar{r} = (x+y)\,dx + (2x-z)dy + (y+z)dz$

$$\bar{F}.d\bar{r} = [(x+y)i + (2x-z)j + (y+z)k].[i\,dx + j\,dy + k\,dz]$$

$$F = (x+y)i + (2x-z)j + (y+z)k.$$

WKT stokes theorem. $\oint \bar{F}.d\bar{r} = \int_s \text{Curl}\,\bar{F}.\bar{n}\,ds$.

$$\text{Curl}\,\bar{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$\bar{F} = F_1 i + F_2 j + F_3 k$

$F_1 = x+y$

$F_2 = 2x-z$

$F_3 = y+z$

$$= \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x+y & 2x-z & y+z \end{vmatrix}$$

$$= i\left[\frac{\partial}{\partial y}(y+z) - \frac{\partial}{\partial z}(2x-z)\right] - j\left[\frac{\partial}{\partial x}(y+z) - \frac{\partial}{\partial z}(x+y)\right] + k\left[\frac{\partial}{\partial x}(2x-z) - \frac{\partial}{\partial y}(x+y)\right]$$

$$\text{Curl}\,\bar{F} = 2i + k$$

An equation of the plane through $A(2,0,0)$ $B(0,3,0)$ $C(0,0,6)$ is

$$\frac{x}{2} + \frac{y}{3} + \frac{z}{6} = 1 \quad (\text{or}) \quad 3x + 2y + z = 6.$$

Normal to the plane is $\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$

Let $\phi = 3x + 2y + z - 6$

$$\nabla\phi = 3i + 2j + k.$$

Unit normal vector $\bar{n} = \dfrac{\nabla\phi}{|\nabla\phi|} = \dfrac{3i+2j+k}{\sqrt{9+4+1}} = \dfrac{3i+2j+k}{\sqrt{14}}$.

$$\int_c \bar{F}.d\bar{r} = \int_c (x+y)dx + (2x-z)dy + (y+z)dz$$

$$= \int_s \text{Curl}\,\bar{F}.\bar{n}\,ds$$

$$= \int_s (2i+k) \cdot \frac{(3i+2j+k)}{\sqrt{14}}\,ds$$

$$\int_C \bar{F}\cdot d\bar{r} = \frac{1}{\sqrt{14}}(6+1)\int_S ds$$

$$= \frac{7}{\sqrt{14}}\int_S ds$$

$$= \frac{7}{\sqrt{14}}\int_S ds = \frac{7}{\sqrt{14}}(\text{Area of } \Delta ABC)$$

$$\int_C \bar{F}\cdot d\bar{r} = \frac{7}{\sqrt{14}}|AB\times AC|\cdot\frac{1}{2} \quad \text{—— (1)}$$

We have $A(2,0,0)$ $B(0,3,0)$ $C(0,0,6)$

$AB = (-2,3,0)$ $\quad AC = (-2,0,6)$

$\overline{AB} = -2i+3j$ $\quad \overline{AC} = -2i+6k.$

$$AB\times AC = \begin{vmatrix} i & j & k \\ -2 & 3 & 0 \\ -2 & 0 & 6 \end{vmatrix}$$

$$AB\times AC = i(18) - j(-12) + k(6)$$

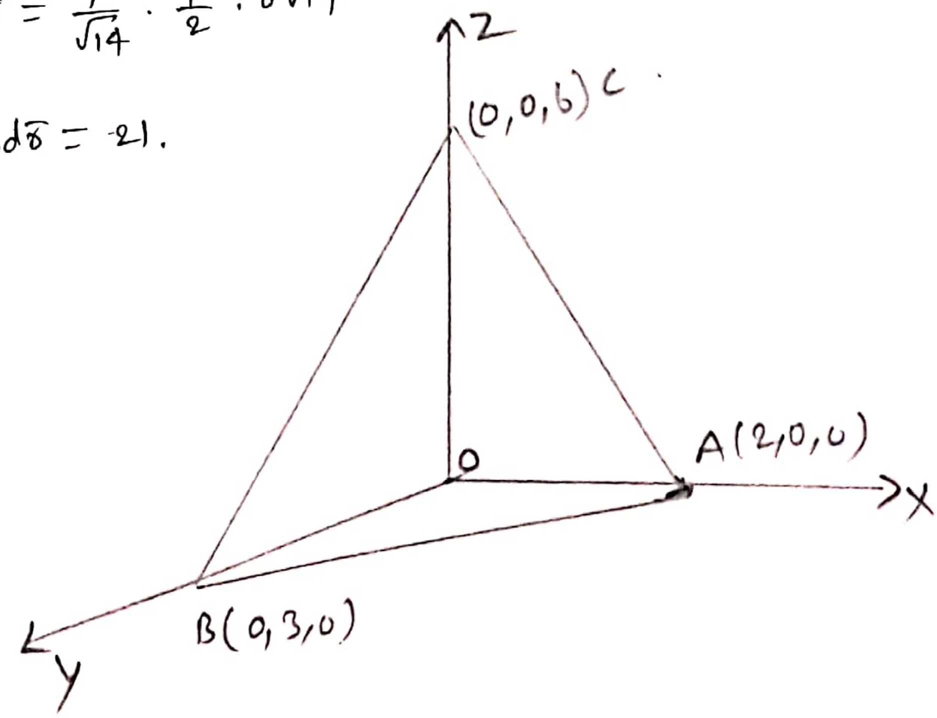
$$AB\times AC = 6(3i+2j+k)$$

$$|AB\times AC| = 6\sqrt{9+4+1} = 6\sqrt{14}. \quad \text{—— (2)}$$

From (1) and (2)

$$\int_C \bar{F}\cdot d\bar{r} = \frac{7}{\sqrt{14}}\cdot\frac{1}{2}\cdot 6\sqrt{14}$$

$$\int_C \bar{F}\cdot d\bar{r} = 21.$$



Volume Integrals :-

Consider a closed surface in space enclosing a volume V. Then, integrals of the form $\iiint_V \bar{F}\,dv$ and $\iiint_V \phi\,dv$ [$\bar{F}$ is a vector function, ϕ is a scalar function] are examples of volume integrals.

Expression of volume integral as the limit of a sum :-

Let $\bar{F}$ be a continuous vector function. Let S be a surface enclosing the region D. Divide this region D into a finite number of subregions $D_1, D_2, D_3 \ldots D_n$.

Let ΔV_i be the volume of the subregion D_i enclosing any point whose position vector is r_i.

Consider the sum $V = \sum_{i=1}^{n} \bar{F}(r_i)\,\Delta V_i$

The limit of this sum as $n \to \infty$ such that $\Delta V_i \to 0$ is called the volume integral of $\bar{F}$ over D and is denoted by $\iiint_D \bar{F}\,dv$.

If $\bar{F} = F_1(x,y,z)\,i + F_2(x,y,z)\,j + F_3(x,y,z)\,k$. So that $dv = dx\,dy\,dz$

$$\iiint_D \bar{F}\,dv = i\iiint_D F_1(x,y,z)\,dx\,dy\,dz + j\iiint_D F_2(x,y,z)\,dx\,dy\,dz + k\iiint_D F_3(x,y,z)\,dx\,dy\,dz$$

→ Evaluate $\iiint_V \phi \, dv$ taken over the rectangular parallelopiped $0 \le x < a$, $0 \le y < b$, $0 \le z < c$ and $\phi = 2(x+y+z)$.

Sol:- Given that $\phi = 2(x+y+z)$

Given that rectangular parallelopiped $0 \le x < a$, $0 \le y < b$, $0 \le z < c$.

$\therefore I = \iiint \phi \, dv$. $\quad [\because dv = dz\,dy\,dx]$

$$I = \int_{x=0}^{x=a} \int_{y=0}^{y=b} \int_{z=0}^{z=c} [2(x+y+z)\,dz]\,dy\,dx.$$

$$= \int_{x=0}^{x=a} \int_{y=0}^{y=b} 2\left[xz + yz + \frac{z^2}{2}\right]_{z=0}^{z=c} dy\,dx.$$

$$= \int_{x=0}^{x=a} \left\{\int_{y=0}^{y=b} [2cx + 2cy + c^2]\,dy\right\} dx$$

$$= \int_{x=0}^{x=a} \left[2cxy + cy^2 + c^2y\right]_{y=0}^{y=b} dx.$$

$$= \int_{x=0}^{x=a} [2bcx + cb^2 + c^2b]\,dx$$

$$= \left[2bc\frac{x^2}{2} + cb^2x + c^2bx\right]_{x=0}^{x=a}$$

$$= a^2bc + ab^2c + abc^2$$

$$= abc(a+b+c)$$

→ If $\bar{F} = (2x^2-3z)i - 2xy\,j - 4x\,k$ evaluate $\iiint_V \nabla\cdot\bar{F}\,dv$ where V is the closed region bounded by the planes $x=0,\ y=0,\ z=0$ and $2x+2y+z=4$.

Sol:- Given that $\bar{F} = (2x^2-3z)i - 2xy\,j - 4x\,k$, $\bar{F} = F_1 i + F_2 j + F_3 k$.

We have to find $\int_V \nabla\cdot\bar{F}\,dv$

$$\nabla\cdot\bar{F} = \operatorname{div}\bar{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$F_1 = 2x^2-3z \quad F_2 = -2xy \quad F_3 = -4x.$$

$$\nabla\cdot\bar{F} = \frac{\partial}{\partial x}(2x^2-3z) + \frac{\partial}{\partial y}(-2xy) + \frac{\partial}{\partial z}(-4x)$$

$$\nabla\cdot\bar{F} = 4x - 2x = 2x$$

Given that the region bounded by the planes $x=0,\ y=0,\ z=0$ and $2x+2y+z=4$.

We have $2x+2y+z=4$ — ① $\Rightarrow z = 4-2x-2y$.

∴ z limits $z=0$, $z=4-2x-2y$.

Given $2x+2y+z=4$

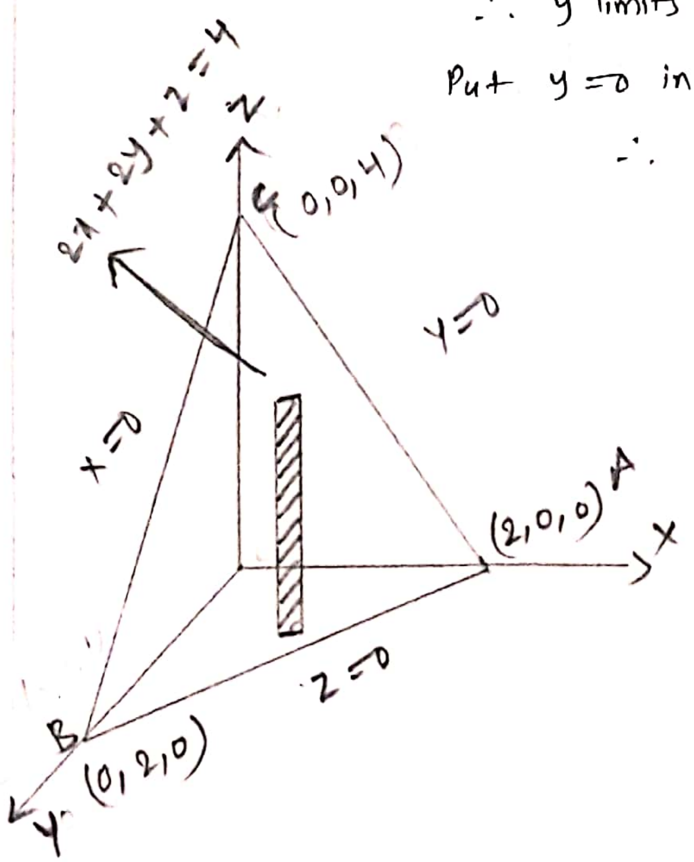
Put $z=0$, $x+y=2$ — ②

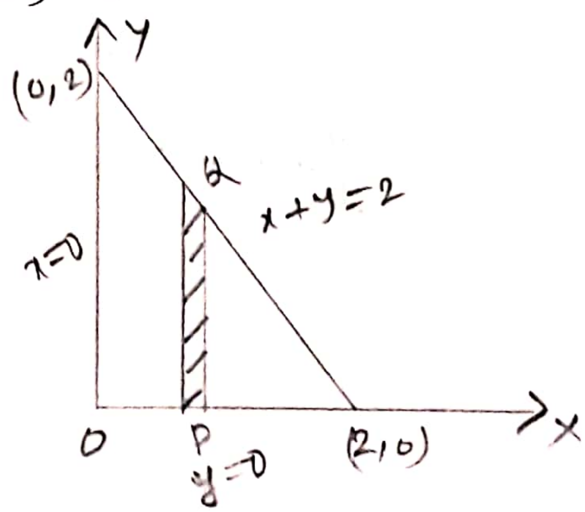
$y = 2-x$

∴ y limits $y=0$ $y=2-x$.

Put $y=0$ in ②, we get $x=2$

∴ x limits $x=0$ $x=2$.





$$\iiint_V \nabla\cdot F\, dv = \int_{x=0}^{x=2}\int_{y=0}^{y=2-x}\int_{z=0}^{z=4-2x-2y} 2x\, dz\,dy\,dx .$$

$$= \int_{x=0}^{x=2}\int_{y=0}^{y=2-x}\left[\int_{z=0}^{z=4-2x-2y} 2x\ dz\right] dy\ dx$$

$$= \int_{x=0}^{x=2}\int_{y=0}^{y=2-x} \left[2xz\right]_{z=0}^{z=4-2x-2y} dy\, dx$$

$$= \int_{x=0}^{x=2}\int_{y=0}^{y=2-x} 2x(4-2x-2y)\,dy\,dx$$

$$= \int_{x=0}^{x=2}\left[\int_{y=0}^{y=2-x}\left[8x-4x^2-4xy\right] dy\right] dx$$

$$= \int_{x=0}^{x=2}\left[8xy-4x^2y-2xy^2\right]_{y=0}^{y=2-x} dx .$$

$$= \int_{x=0}^{x=2}\left[8x(2-x)-4x^2(2-x)-2x(2-x)^2\right] dx$$

$$= \int_{x=0}^{x=2}\left[8x-8x^2+2x^3\right] dx$$

$$= \left[4x^2-\frac{8x^3}{3}+\frac{2x^4}{4}\right]_{x=0}^{x=2}$$

$$= 16-\frac{64}{3}+8$$

$$\iiint_V \nabla\cdot F\, dv = \frac{8}{3}$$

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→ If $\bar{F} = (2x^2-3z)i - 2xy\,j - 4x\,k$ then evaluate $\int_V \nabla\times\bar{F}\,dv$ where V is the closed region bounded by $x=0,\ y=0,\ z=0$ $2x+2y+z=4$.

Sol:- Given that $\bar{F} = (2x^2-3z)i - 2xy\,j - 4x\,k$. $\bar{F} = F_1 i + F_2 j + F_3 k$.

$$\text{Curl}\,\bar{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2x^2-3z & -2xy & -4x \end{vmatrix}$$

$$= i\left[\frac{\partial}{\partial y}(-4x) - \frac{\partial}{\partial z}(-2xy)\right] - j\left[\frac{\partial}{\partial x}(-4x) - \frac{\partial}{\partial z}(2x^2-3z)\right] + k\left[\frac{\partial}{\partial x}(-2xy) - \frac{\partial}{\partial y}(2x^2-3z)\right]$$

$$\text{Curl}\,\bar{F} = j - 2y\,k.$$

Given that the region bounded by the planes $x=0, y=0, z=0$ and $2x+2y+z=4$.

We have $2x+2y+z=4$ —(1) $\Rightarrow z = 4-2x-2y$

$\therefore$ z limits $z=0,\ z=4-2x-2y$

Given $2x+2y+z=4$

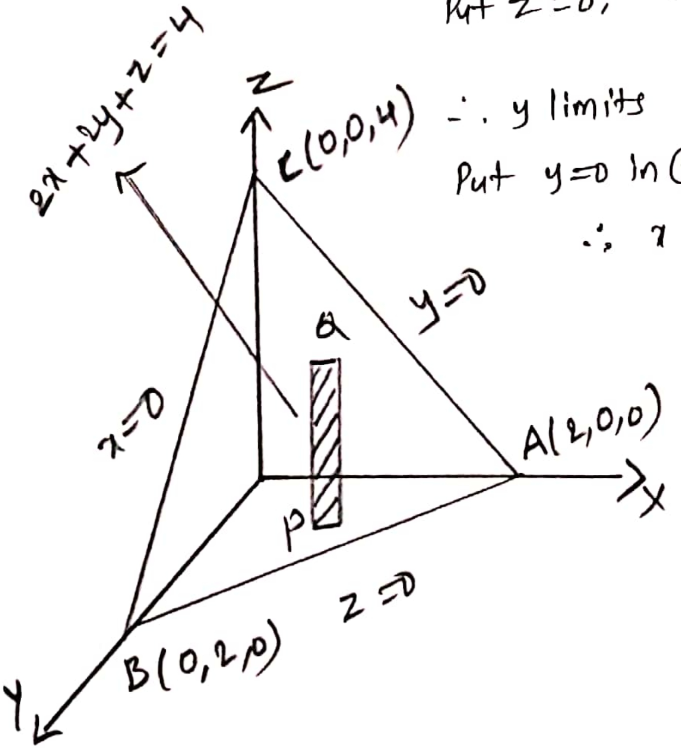
Put $z=0$, $x+y=2$ —(2)

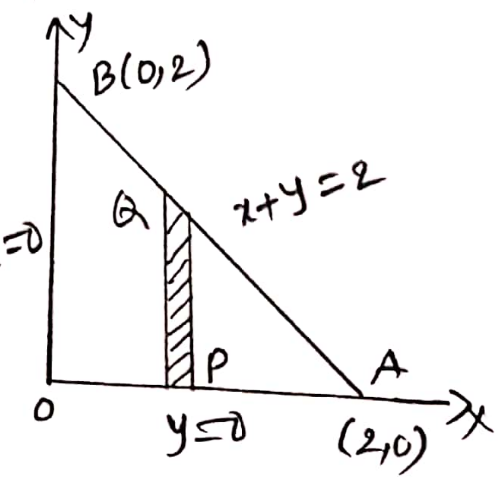
$\Rightarrow y = 2-x$.

$\therefore$ y limits $y=0,\ y=2-x$

Put $y=0$ in (2), we get $x=2$

$\therefore$ x limits $x=0,\ x=2$





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$$\int_V \nabla\times\bar{F}\,dv = \iiint_V (j-2yk)\,dx\,dy\,dz$$

$$= \int_{x=0}^{x=2}\int_{y=0}^{y=2-x}\int_{z=0}^{z=4-2x-2y} (j-2yk)\,dz\,dy\,dx.$$

$$= \int_{x=0}^{x=2}\int_{y=0}^{y=2-x} (j-2yk)\,[z]_{z=0}^{z=4-2x-2y}\,dy\,dx$$

$$= \int_{x=0}^{x=2}\int_{y=0}^{y=2-x} (j-2yk)(4-2x-2y)\,dy\,dx.$$

$$= \int_{x=0}^{x=2}\int_{y=0}^{y=2-x} \left\{j\left[(4-2x)-2y\right] - \bar{k}\left[(4-2x)2y-4y^2\right]\right\}dy\,dx$$

$$= \int_{x=0}^{x=2} j\left[(4-2x)y - y^2\right]_{y=0}^{y=2-x} - k\int_{x=0}^{x=2}\left[(4-2x)y^2 - 4\frac{y^3}{3}\right]_{y=0}^{y=2-x} dx.$$

$$= j\int_{x=0}^{x=2} (2-x)^2\,dx - k\int_{x=0}^{x=2} \frac{2}{3}(2-x)^3\,dx.$$

$$= j\left[\frac{(2-x)^3}{-3}\right]_{x=0}^{x=2} - k\,\frac{2}{3}\left[\frac{(2-x)^4}{-4}\right]_{x=0}^{x=2}$$

$$\int_V (\nabla\times\bar{F})\,dv = \frac{8}{3}(j-k)$$

Gauss's Divergence Theorem :-

(Transformation between Surface Integral and Volume Integral)

Let S be a closed surface enclosing a volume V. If $\bar{F}$ is a continuously differentiable vector point function, then

$\int_V \operatorname{div}\bar{F}\, dv = \int_S \bar{F}\cdot\bar{n}\, ds$ where n is the outward drawn normal vector at any point of S.

→ Verify the divergence theorem for $\bar{F} = (4xy)i - y^2 j + (xz)k$ over the cube bounded by $x=0, x=1, y=0, y=1, z=0$ and $z=1$.

Sol:- Given that $\bar{F} = (4xy)i - y^2 j + (xz)k$

Given that the cube bounded by $x=0, x=1, y=0, y=1, z=0$ and $z=1$.

WKT Gauss Divergence Theorem.

$$\int_V \operatorname{div}\bar{F}\, dv = \int_S \bar{F}\cdot\bar{n}\, ds .$$

To evaluate $\int_V \operatorname{div}\bar{F}\, dv$:-

We have $\bar{F} = (4xy)i - y^2 j + (xz)k$ $\quad \bar{F} = F_1 i + F_2 j + F_3 k.$

$F_1 = 4xy \quad F_2 = -y^2 \quad F_3 = xz$

$$\operatorname{div}\bar{F} = \nabla\cdot\bar{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$\operatorname{div}\bar{F} = \frac{\partial}{\partial x}(4xy) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(xz)$$

$$\operatorname{div}\bar{F} = 4y - 2y + x$$

$$\operatorname{div}F = x + 2y .$$

$$\int_V \operatorname{div}\bar{F} = \int_V (x+2y)\, dv \qquad \because dv = dx\,dy\,dz$$

$$= \int_{x=0}^{x=1}\int_{y=0}^{y=1}\left[\int_{z=0}^{z=1}(x+2y)\,dz\right]dy\,dx .$$

$$= \int_{x=0}^{x=1}\int_{y=0}^{y=1}(x+2y)\left[z\right]_{z=0}^{z=1} dy\,dx$$

$$= \int_{x=0}^{x=1}\left[\int_{y=0}^{y=1}(x+2y)\,dy\right]dx.$$

$$= \int_{x=0}^{x=1}\left[xy+y^2\right]_{y=0}^{y=1}dx$$

$$= \int_{x=0}^{x=1}(x+1)\,dx$$

$$= \left[\frac{x^2}{2}+x\right]_{x=0}^{x=1}$$

$$= \frac{1}{2}+1$$

$$\int \operatorname{div}\bar{F}\,dv = \frac{3}{2}.$$

To evaluate $\int_S \bar{F}.\bar{n}\,ds$:-

To evaluate the surface integral divide the closed surface S of the cube into 6 parts.

i.e S_1 : The face ABFG

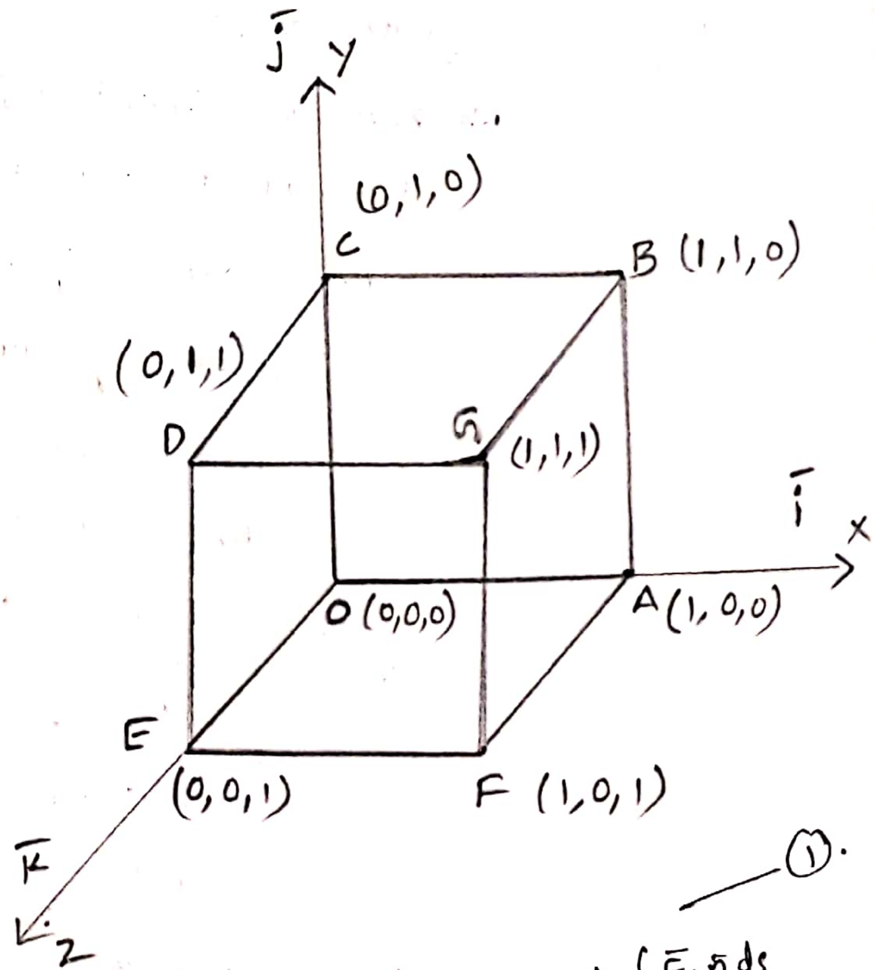
S_2 : The face OCDE

S_3 : The face BCDG.

S_4 : The face OAEF.

S_5 : The face DEFG

S_6 : The face OABC.



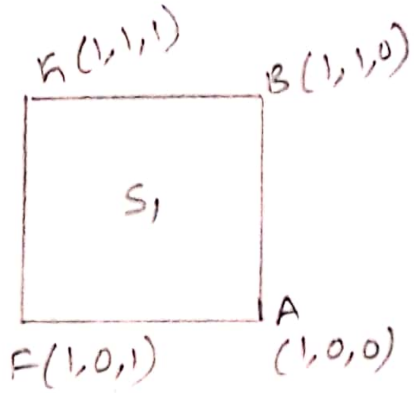
$$\therefore \int_S \bar{F}.\bar{n}\,ds = \int_{S_1}\bar{F}.\bar{n}\,ds + \int_{S_2}\bar{F}.\bar{n}\,ds + \int_{S_3}\bar{F}.\bar{n}\,ds + \dots + \int_{S_6}\bar{F}.\bar{n}\,ds \quad (1).$$

Case (i) To evaluate $\int_{S_1} \bar{F}\cdot\bar{n}\, ds$. (OR) on the face ABFG .

Unit normal vector to S_1 (face ABFG) is $\bar{n} = \bar{i}$

On S_1, $x = 1$,

$ds = dy\,dz$. [$\because$ Face ABFG is ||el to YZ plane]



y varies from 0 to 1 .

$\therefore$ y limits $y=0,\ y=1$.

z varies from 0 to 1

$\therefore$ z limits $z=0,\ z=1$

$\bar{F}\cdot\bar{n} = [4xy\, i - y^2 j + (xz)k]\cdot i$

$\bar{F}\cdot\bar{n} = 4xy$

$\bar{F}\cdot\bar{n} = 4y \quad [\because x = 1]$

$$\int_{S_1} \bar{F}\cdot\bar{n}\, ds = \int_{S_1} 4y\, dy\, dz = \int_{y=0}^{y=1}\int_{z=0}^{z=1} 4y\, dy\, dz$$

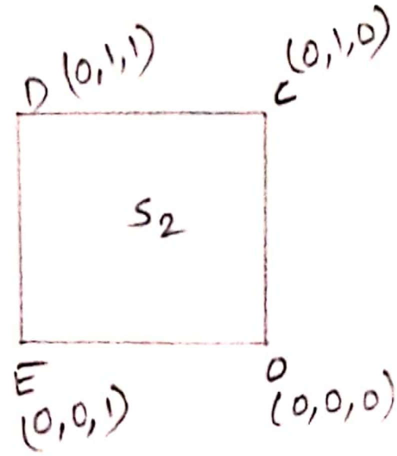
$$= 4\int_{y=0}^{y=1} y\, dy \int_{z=0}^{z=1} dz = 4\left[\frac{y^2}{2}\right]_{y=0}^{y=1} \left[z\right]_{z=0}^{z=1}$$

$$= 4\left(\frac{1}{2} - 0\right)(1-0)$$

$$\int_{S_1} \bar{F}\cdot\bar{n}\, ds = 2 \quad \text{——} \ (2)$$

Case (ii) :- To evaluate $\int_{S_2} \bar{F}\cdot\bar{n}\, ds$ (OR) on the face OCDE :

Unit normal vector to (S_2 face OCDE) is $\bar{n} = -i$



On S_2, $x = 0$

$ds = dy\,dz$ [$\because$ Face OCDE is in YZ plane]

y varies from 0 to 1 .

$\therefore$ y limits $y=0,\ y=1$

z varies from 0 to 1 .

$\therefore$ z limits $z=0,\ z=1$

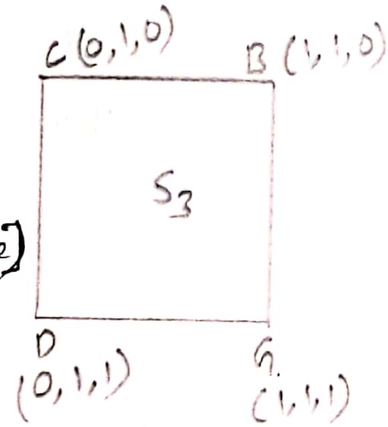
$\bar{F}\cdot\bar{n} = [4xy\, i - y^2 j + (xz)k]\cdot(-i)$

$\bar{F}\cdot\bar{n} = -4xy$

$\bar{F}.\bar{n} = 0 \quad [\because x = 0]$

$$\int_{S_2} \bar{F}.\bar{n}\, ds = 0 \quad \text{——} \quad (3)$$

<u>Case (iii)</u> :- To evaluate $\int_{S_3} \bar{F}.\bar{n}\, ds$ (OR) On the face BCDG.



Unit normal vector to S_3 (face BCDG) is $\bar{n} = j$

On S_3, $y = 1$

$ds = dx\,dz$ [$\because$ Face BCDG is ||lel to xz plane]

x varies from 0 to 1.

$\therefore$ x limits $x = 0$, $x = 1$

z varies from 0 to 1.

$\therefore$ z limits $z = 0$, $z = 1$.

$\bar{F}.\bar{n} = [4xy\, i - y^2 j + (xz) k].j$

$\bar{F}.\bar{n} = -y^2$

$\bar{F}.\bar{n} = -1 \quad [\because y = 1]$

$$\int_{S_3} \bar{F}.\bar{n}\, ds = \int_{S_3} -1.dx\,dz = \int_{x=0}^{x=1} \int_{z=0}^{z=1} -1.dx\,dz$$

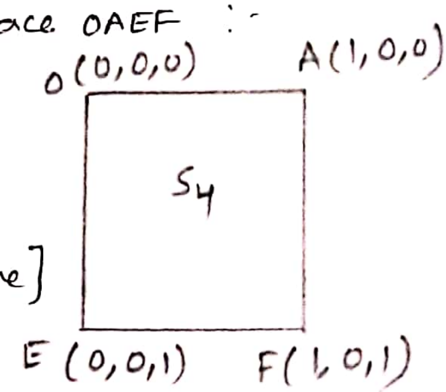
$$= -\int_{x=0}^{x=1} dx \int_{z=0}^{z=1} dz$$

$$= -[x]_{x=0}^{x=1} [z]_{z=0}^{z=1}$$

$$= -(1-0)(1-0)$$

$$\int_{S_3} \bar{F}.\bar{n}\, ds = -1 \quad \text{——} \quad (4)$$

<u>Case (iv)</u> :- To evaluate $\int_{S_4} \bar{F}.\bar{n}\, ds$ (OR) On the face OAEF :-



Unit normal vector to S_4 (face OAEF) is $\bar{n} = -j$

On S_4, $y = 0$.

$ds = dx\,dz$ [$\because$ Face OAEF is in xz plane]

x varies from 0 to 1

$\therefore$ x limits $x=0, x=1$.

z varies from 0 to 1.

$\therefore$ z limits $z=0, z=1$.

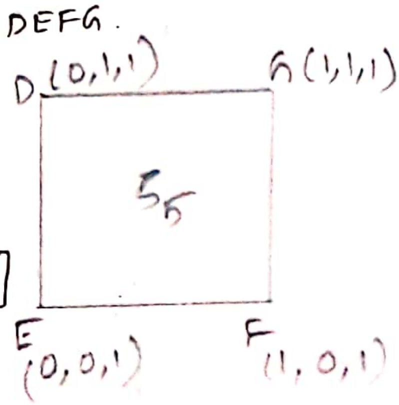
$\bar{F}.\bar{n} = [4xy\,i - y^2 j + (xz)k].(-j)$

$\bar{F}.\bar{n} = y^2$

$\bar{F}.\bar{n} = 0 \quad [\because y=0]$

$\therefore \int_{S_4} \bar{F}.\bar{n}\, ds = 0$ —— ⑤

Case (V) :- To evaluate $\int_{S_5} \bar{F}.\bar{n}\, ds$ (OR) on the face DEFG.



Unit normal vector to S_4 (face DEFG) is $\bar{n} = \bar{k}$

on S_5, $z=1$

$ds = dx\,dy$ [$\because$ Face DEFG is ||lel to xy-plane]

x varies from 0 to 1

$\therefore$ x limits $x=0, x=1$

y varies from 0 to 1

$\therefore$ y limits $y=0$ $y=1$

$\bar{F}.\bar{n} = [4xy\,i - y^2 j + (xz)k].k$

$\bar{F}.\bar{n} = xz$

$\bar{F}.\bar{n} = x \quad [\because z=1]$

$$\int_{S_5} \bar{F}.\bar{n}\, ds = \int_{S_5} x\, dx\, dy = \int_{x=0}^{x=1}\int_{y=0}^{y=1} x\, dx\, dy$$

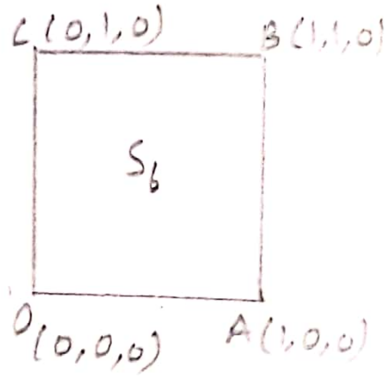
$$= \int_{x=0}^{x=1} x\, dx \int_{y=0}^{y=1} dy$$

$$= \left[\frac{x^2}{2}\right]_{x=0}^{1} \left[y\right]_{y=0}^{y=1}$$

$$= \left(\frac{1}{2}-0\right)(1-0)$$

$\int_{S_5} \bar{F}.\bar{n}\, ds = \frac{1}{2}$ —— ⑥

Case (vi) To evaluate $\int_{S_6} \bar{F}\cdot\bar{n}\, ds$ (OR) on the face OABC.



Unit normal vector to S_6 (face OABC) is $\bar{n} = -\bar{k}$

On S_6, $z = 0$

$ds = dx\,dy$ [$\because$ Face OABC is in xy-plane]

x varies from 0 to 1

$\therefore$ x limits $x=0$, $x=1$

y varies from 0 to 1

$\therefore$ y limits $y=0$, $y=1$

$\bar{F}\cdot\bar{n} = [4xy\,i - y^2 j + (xz)k]\cdot(-k)$

$\bar{F}\cdot\bar{n} = -xz$

$\bar{F}\cdot\bar{n} = 0$ [$\because z=0$]

$\int_{S_6} \bar{F}\cdot\bar{n}\, ds = 0$ —— (7)

Sub. (2) (3) (4) (5), (6) and (7) in (1), we get

$\int_S \bar{F}\cdot\bar{n}\, ds = 2 + 0 - 1 + 0 + \frac{1}{2} + 0$

$\int_S \bar{F}\cdot\bar{n}\, ds = \frac{3}{2}$.

$\therefore$ $\int_V \operatorname{div}\bar{F}\, dv = \int_S \bar{F}\cdot\bar{n}\, ds$

$\therefore$ Gauss Divergence Theorem verified.

Verify Gauss divergence theorem for $f = (x^2-yz)i + (y^2-zx)j + (z^2-xy)k$ taken over the rectangular parallelopiped $0 \le x \le a,\ 0 \le y \le b,\ 0 \le z \le c$.

Sol:- Given that $f = (x^2-yz)i + (y^2-zx)j + (z^2-xy)k$

and the given the rectangular parallelopiped $0 \le x \le a,\ 0 \le y \le b,\ 0 \le z \le c$.

We know that the Gauss Divergence theorem.

$$\int_S F.n\, ds = \iiint_V \nabla.F\, dv.$$

To evaluate $\iiint_V \nabla.F\, dv$:-

$$\nabla.F = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}.$$

$$\nabla.F = \frac{\partial}{\partial x}(x^2-yz) + \frac{\partial}{\partial y}(y^2-zx) + \frac{\partial}{\partial z}(z^2-xy)$$

$$\nabla.F = 2x + 2y + 2z.$$

$$\iiint_V \nabla.F\, dv = \int_{x=0}^{x=a}\int_{y=0}^{y=b}\int_{z=0}^{z=c} 2(x+y+z)\, dz\, dy\, dx.$$

$$= 2\int_{x=0}^{x=a}\int_{y=0}^{y=b}\left[xz + yz + \frac{z^2}{2}\right]_{z=0}^{z=c} dy\, dx$$

$$= 2\int_{x=0}^{x=a}\int_{y=0}^{y=b} cx + cy + \frac{c^2}{2}\Big] dy\, dx.$$

$$= 2\int_{x=0}^{x=a}\left[cxy + c\frac{y^2}{2} + \frac{c^2}{2}y\right]_{y=0}^{y=b} dx$$

$$= 2\int_{x=0}^{x=a}\left[bcx + \frac{cb^2}{2} + \frac{bc^2}{2}\right] dx$$

$$= 2\left[bc\frac{x^2}{2} + \frac{cb^2}{2}x + \frac{bc^2}{2}x\right]_{x=0}^{x=a}$$

$$= 2\left[\frac{bca^2}{2} + \frac{abc^2}{2} + \frac{acb^2}{2}\right]$$

$$\iiint_V (\nabla \cdot f)\, dv = abc(a+b+c) \quad \text{--- (1)}$$

To evaluate $\int_S \bar{F} \cdot \bar{n}\, ds$:-

S is the surface of the rectangular parallelopiped given by

$0 \le x \le a,\ 0 \le y \le b,\ 0 \le z \le c.$

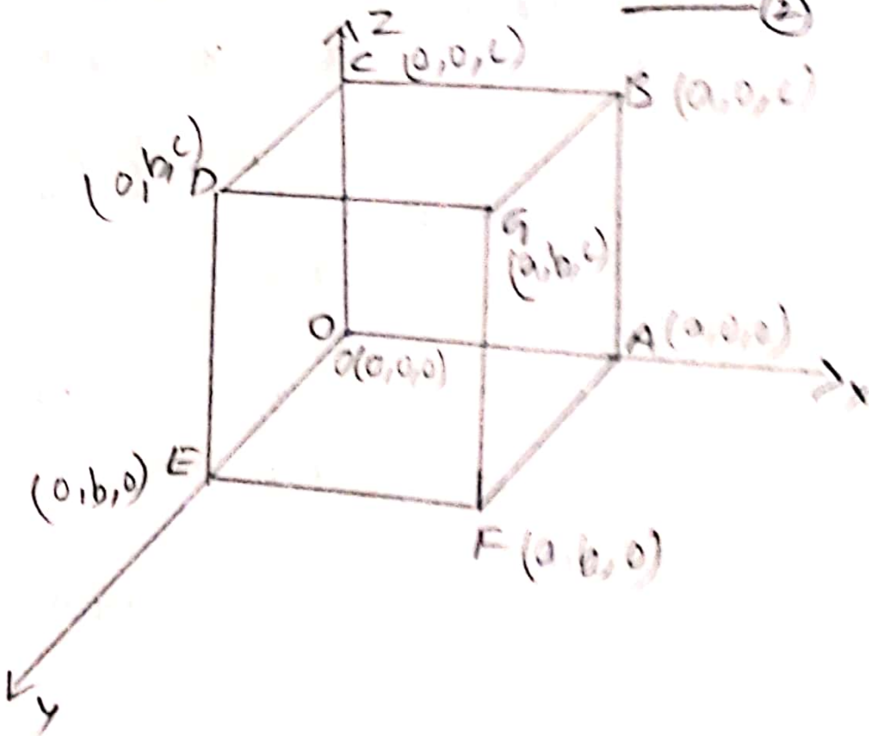
We note that the boundary surface S of the given rectangular parallelopiped is made up of the following six faces.

S_1 : OABC (xz plane) $\qquad$ S_2 : DEGF (opposite to xz plane)

S_3 : OCDE (yz plane) $\qquad$ S_4 : ABFG (opposite to yz plane)

S_5 : OAEF (xy plane) $\qquad$ S_6 : BCDG (opposite to xy plane)

$$\int_S \bar{F} \cdot \bar{n}\, ds = \int_{S_1} \bar{F} \cdot \bar{n}\, ds + \int_{S_2} \bar{F} \cdot \bar{n}\, ds + \int_{S_3} \bar{F} \cdot \bar{n}\, ds + \cdots + \int_{S_6} \bar{F} \cdot \bar{n}\, ds \quad \text{--- (2)}$$



(i) To evaluate $\int_{S_1} \bar{F}\cdot\bar{n}\, ds$:-

On the face OABC we have $y=0$, $0\le x\le a$ and $0\le z\le c$.

The unit outward drawn normal to the face OABC is $\bar{n}=-j$

$$\int_{S_1} \bar{F}\cdot\bar{n}\, ds = \int_{x=0}^{x=a}\int_{z=0}^{z=c} \left[(x^2-0z)i + (0.zx)j + (z^2-x.0)k\right]\cdot(-j)\, dz\, dx.$$

$$= \int_{x=0}^{x=a}\int_{z=0}^{z=c} zx\, dz\, dx$$

$$= \int_{x=0}^{x=a} x\, dx \left[\frac{z^2}{2}\right]_{z=0}^{z=c}$$

$$= \frac{c^2}{2}\int_{x=0}^{x=a} x\, dx = \frac{c^2}{2}\left[\frac{x^2}{2}\right]_{x=0}^{x=a}$$

$$= \frac{a^2c^2}{4}. \quad \text{——— (3)}$$

(ii) To evaluate $\int_{S_2} \bar{F}\cdot\bar{n}\, ds$:—

On the face DEFG we have $y=b$, $0\le x\le a$, $0\le z\le c$.

The unit outward drawn normal to the face DEFG is $\bar{n}=\bar{j}$

$$\int_{S_2} \bar{F}\cdot\bar{n}\, ds = \int_{x=0}^{x=a}\int_{z=0}^{z=c} \left[(x^2-bz)i + (b^2-zx)j + (z^2-bx)k\right]\cdot j\, dz\, dx.$$

$$= \int_{x=0}^{x=a}\int_{z=0}^{z=c} (b^2-zx)\, dz.dx.$$

$$= \int_{x=0}^{x=a}\left[b^2z - \frac{z^2x}{2}\right]_{z=0}^{z=c} dx = \int_{x=0}^{x=a}\left[cb^2 - \frac{c^2x}{2}\right] dx$$

$$= \left[cb^2x - \frac{c^2x^2}{4}\right]_{x=0}^{x=a}$$

$$= ab^2c - \frac{a^2c^2}{4}. \quad \text{——— (4)}$$

(iii) To evaluate $\int_{S_3} \bar{F}.\bar{n}\, ds$:—

On the face OCDE we have $x=0$, $0 \le y \le b$, $0 \le z \le c$.

The unit outward drawn normal to the face OCDE is $\bar{n} = -i$

$$\int_{S_3} \bar{F}.\bar{n}\, ds = \int_{y=0}^{y=b}\int_{z=0}^{z=c} \left[(0-yz)i + (y^2-0.z)j + (z^2-0.y)k\right].(-i)\, dy\, dz$$

$$= \int_{y=0}^{y=b}\int_{z=0}^{z=c} yz\, dy\, dz$$

$$= \int_{y=0}^{y=b} y\, dy \left[\frac{z^2}{2}\right]_{z=0}^{z=c}$$

$$= \frac{c^2}{2}\int_{y=0}^{y=b} y\, dy = \frac{c^2}{2}\left[\frac{y^2}{2}\right]_{y=0}^{y=b}$$

$$\int_{S_3} \bar{F}.\bar{n}\, ds = \frac{b^2c^2}{4}. \quad \text{—— (5)}$$

(iv) To evaluate $\int_{S_4} \bar{F}.\bar{n}\, ds$:-

On the face ABFG we have $x=a$, $0 \le y \le b$, $0 \le z \le c$.

The unit outward drawn normal to the face ABFG is $\bar{n} = \bar{i}$

$$\int_{S_4} \bar{F}.\bar{n}\, ds = \int_{y=0}^{y=b}\int_{z=0}^{z=c} \left[(a^2-yz)i + (y^2-az)j + (z^2-ay)k\right].i\, dy\, dz$$

$$= \int_{y=0}^{y=b}\int_{z=0}^{z=c} (a^2-yz)\, dy\, dz$$

$$= \int_{y=0}^{y=b} \left[a^2z - \frac{yz^2}{2}\right]_{z=0}^{z=c} dy = \int_{y=0}^{y=b} \left[ca^2 - \frac{c^2y}{2}\right] dy$$

$$= \left[ca^2y - \frac{c^2y^2}{4}\right]_{y=0}^{y=b}$$

$$= a^2bc - \frac{c^2b^2}{4}. \quad \text{—— (6)}$$

(vi) To evaluate $\int\int_{S_6} \bar{f}.\bar{n}\, ds$:-

On the face BCDG we have $z=c,\ 0\le x\le a,\ 0\le y\le b$.

The unit outward drawn normal to the face BCDG is $\bar{n}=\bar{k}$

$$\int\int_{S_6} \bar{f}.\bar{n}\, ds = \int_{x=0}^{x=a}\int_{y=0}^{y=b} \left[(x^2-cy)i+(y^2-cz)j+(c^2-xy)k\right].\bar{k}\, dx\, dy$$

$$= \int_{x=0}^{x=a}\int_{y=0}^{y=b} (c^2-xy)\, dx\, dy.$$

$$= \int_{x=0}^{x=a}\left[c^2y-\frac{xy^2}{2}\right]_{y=0}^{y=b} dx.$$

$$= \int_{x=0}^{x=a}\left[bc^2-\frac{b^2x}{2}\right] dx$$

$$= \left[bc^2x-\frac{b^2x^2}{4}\right]_{x=0}^{x=a}$$

$$= abc^2-\frac{a^2b^2}{4}. \quad \text{——} \ (7)$$

(VI) To evaluate $\int\int_{S_5} \bar{f}.\bar{n}\, ds$:—

On the face OAEF we have $z=0,\ 0\le x\le a,\ 0\le y\le b$.

The unit outward drawn normal to the face OAEF is $\bar{n}=-\bar{k}$

$$\int\int_{S_5} \bar{f}.\bar{n}\, ds = \int_{x=0}^{x=a}\int_{y=0}^{y=b} \left[(x^2-0.y)i+(y^2-0.x)j+(0-xy)k\right].(-\bar{k})\, dx\, dy$$

$$= \int_{x=0}^{x=a}\int_{y=0}^{y=b} xy\, dx\, dy$$

$$= \int_{x=0}^{x=a} x\,dx\ \left[\frac{y^2}{2}\right]_{y=0}^{y=b}$$

$$= \frac{b^2}{2}\left[\frac{x^2}{2}\right]_{x=0}^{x=a}$$

$$= \frac{a^2b^2}{4} \quad \text{———} \quad (8)$$

Sub. (3).(4) ... and (8) in (2), we get

$$\int_S \bar{F}.\bar{n}\,ds = \frac{a^2c^2}{4} + a^2bc - \frac{a^2c^2}{4} + \frac{b^2c^2}{4} + a^2bc - \frac{c^2b^2}{4} + abc^2 - \frac{a^2b^2}{4} + \frac{a^2b^2}{4}$$

$$\int_S \bar{F}.\bar{n}\,ds = abc\,(a+b+c) \quad \text{———} \quad (9)$$

$\therefore$ From (1) and (9).

$$\int_S \bar{F}.\bar{n}\,ds = \iiint_V \nabla.\bar{F}\,dv$$

$\therefore$ Gauss Divergence theorem verified.

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Verify the divergence theorem for $F = 4xy\,i - y^2 j + xz\,k$ over the cube bounded by $x=0,\ x=1,\ y=0\ y=1,\ z=0$ and $z=1$

Sol: Given that $F = 4xy\,i - y^2 j + xz\,k$.

The cube is bounded by $x=0\ x=1\ y=0\ y=1\ z=0$ and $z=1$

By Gauss Divergence theorem $\iint_S F.n\,ds = \iiint_V \nabla.F\,dv$.

To Evaluate $\iiint_V \nabla.F\,dv$:—

$$\nabla.F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$= \frac{\partial(4xy)}{\partial x} + \frac{\partial(-y^2)}{\partial y} + \frac{\partial(xz)}{\partial z}$$

$$= 4y - 2y + x = x + 2y$$

$$\nabla.F = x + 2y.$$

$$\iiint_V \nabla.F\,dv = \int_{x=0}^{x=1}\int_{y=0}^{y=1}\int_{z=0}^{z=1} (x+2y)\,dz\,dy\,dx.$$

$$= \int_{x=0}^{x=1}\int_{y=0}^{y=1} (x+2y)\,dy\,dx.\left[z\right]_{z=0}^{z=1}$$

$$= \int_{x=0}^{x=1}\int_{y=0}^{y=1} (x+2y)\,dy.\,dx$$

$$= \int_{x=0}^{x=1} \left[xy + y^2\right]_{y=0}^{y=1} dx.$$

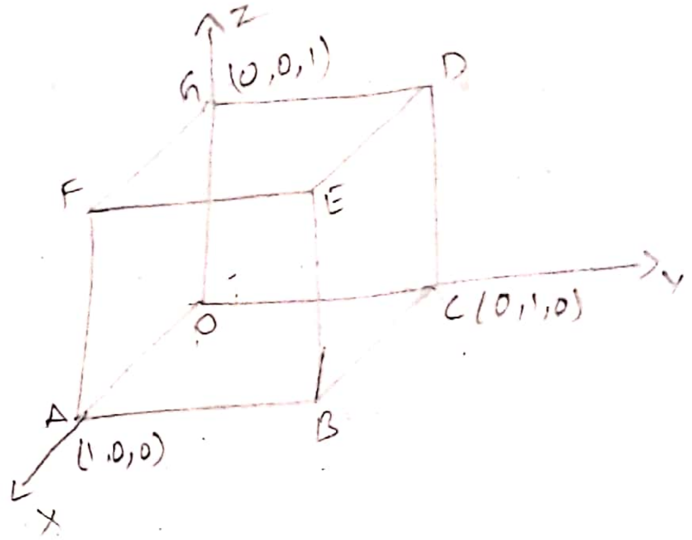
$$= \int_{x=0}^{x=1} (x+1)\,dx. = \left[\frac{(x+1)^2}{2}\right]_{x=0}^{x=1}$$

$$= 2 - \frac{1}{2} = \frac{3}{2}.$$

$$\iiint_V \nabla.F\,dv = \frac{3}{2} \quad \text{—} (1)$$

To evaluate $\iint_S F.n\,ds$:—

To evaluate the surface integral divide the closed surface S of the cube into 6 parts.



The Surface S contains 6 faces.

S_1 : The face. BCDE

S_2 : The face OAFG

S_3 : The face ABEF.

S_4 : The face OCDG

S_5 : The face DEFG

S_6 : The face OABC.

The surface integral $\iint_S \bar{F}.\bar{n}\,ds$ is equal to the sum of the surface integrals on the above 6 faces.

$$\therefore \iint_S \bar{F}.\bar{n}\,ds = \iint_{S_1} \bar{F}.\bar{n}\,ds + \iint_{S_2} \bar{F}.\bar{n}\,ds + \cdots + \iint_{S_6} \bar{F}.\bar{n}\,ds. \quad \text{—— (2)}$$

The unit outward normals to these faces are $j, -j, i, -i, k$ and $-k$ respectively.

(i) on S_1. The face. BCDE :

$\bar{n} = \bar{j} \qquad y = 1. \qquad ds = dx\,dz.$

$$\bar{F}.\bar{n} = [(4xy)\bar{i} - y^2 j + yz k].j = -y^2 = -1 \quad (\because y=1).$$

$$\iint_{S_1} \bar{F}.\bar{n}\,ds = \int_{x=0}^{x=1}\int_{z=0}^{z=1} -1\,dx\,dz$$

$$= \int_{x=0}^{x=1} [-z]_0^1\,dx = \int_{x=0}^{x=1} -dx.$$

$$= [-x]_0^1 = -1.$$

$$\therefore \iint_{S_1} \bar{F}.\bar{n}\,ds = -1 \quad \text{—— (3)}.$$

(ii) on S_2 The face. OAFG.

$\bar{n} = -j, \quad y=0, \quad ds = dx\,dz.$

$F.\bar{n} = -y^2 = 0.$

$\therefore \iint_{S_2} F.\bar{n}\,ds = 0. \quad — (4)$

(iii) on S_3 The face ABEF.

$\bar{n} = \bar{i}, \quad x=1 \quad ds = dy\,dz.$

$F.\bar{n} = [(4xy)i - y^2 j + (z2)k].i = 4xy$

$F.\bar{n} = 4y \quad (\because x=1)$

$$\iint_{S_3} F.\bar{n}\,ds = \int_{y=0}^{y=1}\int_{z=0}^{z=1} 4y\,dy\,dz.$$

$$= \int_{y=0}^{y=1} 4y\,dy\,[z]_{z=0}^{z=1}$$

$$= \int_{y=0}^{y=1} 4y\,dy$$

$$= \left[2y^2\right]_{y=0}^{y=1} = 2.$$

$\iint_{S_3} F.\bar{n}\,ds = 2. — (5)$

(iv) on S_4 The face. OCDG.

$\bar{n} = -i \quad x=0 \quad ds = dy\,dz.$

$F.\bar{n} = 4xy$

$F.\bar{n} = 0 \quad (\because x=0)$

$\therefore \iint_{S_4} F.\bar{n}\,ds = 0. — (6).$

on S_5. The face DEFG :-

$$\bar{n} = \bar{k} \qquad z = 1 \qquad ds = dx\,dy.$$

$$\bar{F}.\bar{n} = [(4xy)i - y^2 j + (xz)k].\bar{k} = xz$$

$$\bar{F}.\bar{n} = x \qquad (\because z = 1)$$

$$\iint_{S_5} \bar{F}.\bar{n}\,ds = \int_{x=0}^{x=1}\int_{y=0}^{y=1} x\,dx\,dy$$

$$= \int_{x=0}^{x=1} x\,dx\,[y]_{y=0}^{y=1}$$

$$= \int_{x=0}^{x=1} x\,dx = \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2}$$

$$\iint_{S_5} \bar{F}.\bar{n}\,ds = \frac{1}{2}. \quad \text{—(7)}$$

On S_6 The face OABC.

$$\bar{n} = -\bar{k} \qquad z = 0 \qquad ds = dx\,dy.$$

$$\bar{F}.\bar{n} = xz$$

$$\bar{F}.\bar{n} = 0 \qquad \because z = 0.$$

$$\iint_{S_6} \bar{F}.\bar{n}\,ds = 0. \quad \text{—(8)}$$

Sub. (3), (4) .... (8) in (2), we get

$$\iint_S \bar{F}.\bar{n}\,ds = -1 + 0 + 2 + 0 + \frac{1}{2} + 0. = \frac{3}{2}$$

$$\iint_S \bar{F}.\bar{n}\,ds = \frac{3}{2} \quad \text{—(9).}$$

From (1) and (9).

$$\iiint_V \nabla.F\,dv = \iint_S \bar{F}.\bar{n}\,ds.$$

$\therefore$ Gauss Divergence theorem verified.

→ Verify Gauss Divergence Theorem for $\bar{F} = x^2 i + y^2 j + z^2 k$ over the surface S of the solid cut off by the plane $x+y+z=a$ in the first octant.

sol:- Given that $F = x^2 i + y^2 j + z^2 k$

Given that the surface S of the solid cut off by the plane $x+y+z=a$ in the first octant.

WKT Gauss Divergence Theorem

$$\int_V \operatorname{div} \bar{F}\, dv = \int_S \bar{F}\cdot\bar{n}\, ds.$$

To evaluate $\int_V \operatorname{div}\bar{F}\, dv$:-

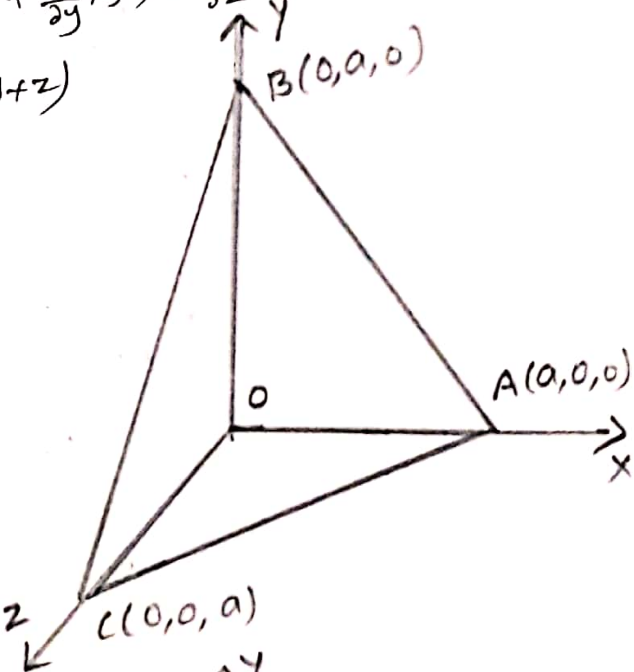
We have $\bar{F} = x^2 i + y^2 j + z^2 k$ $\qquad \bar{F} = F_1 i + F_2 j + F_3 k$

$F_1 = x^2 \quad F_2 = y^2 \quad F_3 = z^2$

WKT $\operatorname{div}\bar{F} = \nabla\cdot F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$

$\nabla\cdot\bar{F} = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(z^2)$

$\nabla\cdot\bar{F} = 2(x+y+z)$



We have the plane $x+y+z=a$

$\Rightarrow z = a-x-y$

z limits $z=0,\ z=a-x-y$

The projection of S in xy-plane is ΔOAB

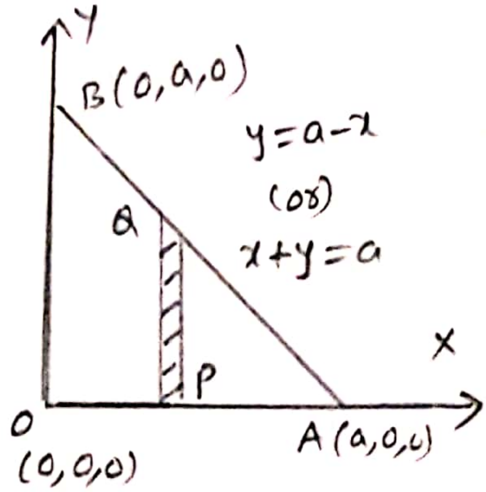
Here x varies from 0 to a

$\therefore$ x limits $x=0,\ x=a$.

For each x, y varies from a point P on x-axis $(y=0)$ to a point Q on the line.

$x+y=a$ i.e $y=a-x$

$\therefore$ y limits $y=0,\ y=a-x$



$$\iiint_V \operatorname{div}\vec{F}\,dv = \iiint_V 2(x+y+z)\,dz\,dy\,dx$$

$$= 2\int_{x=0}^{x=a}\int_{y=0}^{y=a-x}\int_{z=0}^{z=a-x-y}\left[(x+y)+z\right]dz\,dy\,dx$$

$$= 2\int_{x=0}^{x=a}\int_{y=0}^{y=a-x}\left[z(x+y)+\frac{z^2}{2}\right]_{z=0}^{z=a-x-y} dy\,dx.$$

$$= 2\int_{x=0}^{x=a}\int_{y=0}^{y=a-x}(a-x-y)\left[(x+y)+\frac{a-x-y}{2}\right]dy\,dx.$$

$$= \int_{x=0}^{x=a}\int_{y=0}^{y=a-x}(a-x-y)(a+x+y)\,dy\,dx$$

$$= \int_{x=0}^{x=a}\int_{y=0}^{y=a-x}\left[a^2-(x+y)^2\right]dy\,dx$$

$$= \int_{x=0}^{x=a}\int_{y=0}^{y=a-x}\left[a^2-x^2-y^2-2xy\right]dy\,dx$$

$$= \int_{x=0}^{x=a}\left[a^2y-x^2y-\frac{y^3}{3}-xy^2\right]_{y=0}^{y=a-x}dx$$

$$= \int_{x=0}^{x=a}(a-x)\frac{(2a^2-x^2-ax)}{3}\,dx.$$

$$= \frac{1}{3}\int_{x=0}^{x=a}\left[2a^3-ax^2-a^2x-2a^2x+x^3+ax^2\right]dx$$

$$= \frac{1}{3}\left[2a^3x-a\frac{x^3}{3}-a^2\cdot\frac{x^2}{2}-a^2x^2+\frac{x^4}{4}+\frac{ax^3}{3}\right]_{x=0}^{x=a}$$

$$\iiint_V \operatorname{div}\vec{F}\,dv = \frac{a^4}{4}.$$

To evaluate $\int_S \bar{F}.\bar{n}\, ds$:-

Let $\phi = x+y+z-a$ be the given plane.

$$\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$

$$\frac{\partial\phi}{\partial x} = 1 \quad \frac{\partial\phi}{\partial y} = 1 \quad \frac{\partial\phi}{\partial z} = 1$$

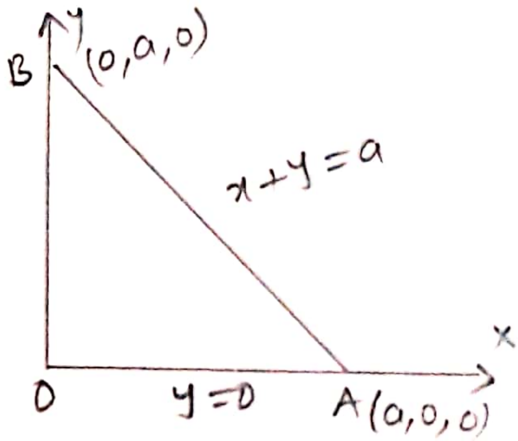
$$\nabla\phi = i+j+k.$$

Unit normal vector $\bar{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{i+j+k}{\sqrt{1+1+1}}$

$$\bar{n} = \frac{i+j+k}{\sqrt{3}}$$

$$\bar{F}.\bar{n} = (x^2 i + y^2 j + z^2 k) . \frac{i+j+k}{\sqrt{3}} = \frac{x^2+y^2+z^2}{\sqrt{3}}.$$

Let R be the projection of S in xy-plane. Which is a ΔOAB.



Here x varies from 0 to a.

$\therefore$ x limits $x=0$ $x=a$.

For each x, y varies 0 to $a-x$

$\therefore$ y limits $y=0$ $y=a-x$.

$$\int_S \bar{F}.\bar{n}\, ds = \iint_R \bar{F}.\bar{n} \frac{dx\,dy}{|\bar{n}.\bar{k}|}$$

$$= \int_{x=0}^{x=a}\int_{y=0}^{y=a-x} \frac{x^2+y^2+z^2}{\sqrt{3}} \frac{dx\,dy}{\frac{1}{\sqrt{3}}}$$

$$= \int_{x=0}^{x=a}\int_{y=0}^{y=a-x} [x^2+y^2+(a-x-y)^2]\, dy\, dx \quad [\because x+y+z=a]$$

$$= \int_{x=0}^{x=a}\int_{y=0}^{y=a-x} [2x^2+2y^2-2ax+2xy-2ay+a^2]\, dy\, dx$$

$$= \int_{x=0}^{x=a} \left[2x^2y + \frac{2}{3}y^3 - 2axy + xy^2 - ay^2 + a^2y\right]_{y=0}^{y=a-x} dx$$

$$= \int_{x=0}^{x=a} \left[2x^2(a-x) + \frac{2}{3}(a-x)^3 - 2ax(a-x) + x(a-x)^2 - a(a-x)^2 + a^2(a-x)\right] dx$$

$$= \int_{x=0}^{x=a} \left[-\frac{5}{3}\cdot x^3 + 3ax^2 - 2a^2x + \frac{2}{3}a^3\right] dx$$

$$= \left[-\frac{5}{3}\cdot\frac{x^4}{4} + ax^3 - a^2x^2 + \frac{a^3 x}{6}\right]_{x=0}^{x=a}$$

$$= -\frac{5a^4}{12} + a^4 - a^4 + \frac{a^4}{6}$$

$$\int_S \bar{F}\cdot\bar{n}\, ds = \frac{a^4}{4}$$

$$\therefore \int_V \operatorname{div}\bar{F}\, dv = \int_S \bar{F}\cdot\bar{n}\, ds.$$

$\therefore$ Gauss Divergence Theorem verified.

→ Use Divergence theorem to evaluate $\iint_S \bar{F}\cdot ds$ where $\bar{F} = x^3 i + y^3 j + z^3 k$ and S is the surface of the sphere $x^2+y^2+z^2 = r^2$.

Sol:- Given that $\bar{F} = x^3 i + y^3 j + z^3 k$, $\bar{F} = F_1 i + F_2 j + F_3 k$.

$F_1 = x^3 \quad F_2 = y^3 \quad F_3 \; z^3$.

We have to find $\iint_S \bar{F}\cdot ds$.

WKT Gauss Divergence theorem

$$\int_S \bar{F}\cdot\bar{n}\,ds = \int_V \mathrm{div}\bar{F}\,dv.$$

$$\mathrm{div}\bar{F} = \nabla\cdot\bar{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$\mathrm{div}\bar{F} = \frac{\partial}{\partial x}(x^3) + \frac{\partial}{\partial y}(y^3) + \frac{\partial}{\partial z}(z^3)$$

$$\mathrm{div}\bar{F} = 3(x^2+y^2+z^2)$$

Given that S is the surface of the sphere $x^2+y^2+z^2 = r^2$.

Changing into spherical polar co ordinates $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$

$z = r\cos\theta$, $dx\,dy\,dz = r^2\sin\theta\,dr\,d\theta\,d\phi$.

r limits $r=0, r=a$ $\quad$ θ limits $\theta=0, \theta=\pi$ $\quad$ ϕ limits $\phi=0, \phi=2\pi$

$$\int_S \bar{F}\cdot\bar{n}\,ds = \int_V \mathrm{div}\bar{F}\,dv = \iiint_V 3(x^2+y^2+z^2)\,dx\,dy\,dz$$

$$= 3\int_{r=0}^{r=a}\int_{\theta=0}^{\theta=\pi}\int_{\phi=0}^{\phi=2\pi} r^2(r^2\sin\theta)\,dr\,d\theta\,d\phi$$

$$= 3\int_{r=0}^{r=a} r^4\,dr \int_{\theta=0}^{\theta=\pi}\sin\theta\,d\theta \int_{\phi=0}^{\phi=2\pi} d\phi$$

$$= 3\left[\frac{r^5}{5}\right]_{r=0}^{r=a}\left[-\cos\theta\right]_{\theta=0}^{\theta=\pi}\left[\phi\right]_{\phi=0}^{\phi=2\pi}$$

$$= 3\left[\frac{a^5}{5}-0\right]\left[-\cos\pi+\cos 0\right]\left[2\pi-0\right]$$

$$= \frac{12\pi a^5}{5}$$

→ Apply Divergence theorem to evaluate $\iint_S (x+z)\,dy\,dz + (y+z)\,dz\,dx + (x+y)\,dx\,dy$ Where S is the surface of the sphere $x^2+y^2+z^2 = 4$.

Sol: Given that $\iint_S (x+z)\,dy\,dz + (y+z)\,dz\,dx + (x+y)\,dx\,dy$

WKT $\int_S \bar{F}.\bar{n}\,ds = \iint_S F_1\,dy\,dz + F_2\,dz\,dx + F_3\,dx\,dy$.

$F_1 = x+z \qquad F_2 = y+z \qquad F_3 = x+y$

WKT Gauss Divergence Theorem.

$$\int_S \bar{F}.\bar{n}\,ds = \int_V \operatorname{div}\bar{F}\,dv.$$

i.e $$\iint_S F_1\,dy\,dz + F_2\,dz\,dx + F_3\,dx\,dy = \iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dx\,dy\,dz$$

$$\frac{\partial F_1}{\partial x} = 1 \qquad \frac{\partial F_2}{\partial y} = 1 \qquad \frac{\partial F_3}{\partial z} = 0$$

Given that S is the surface of the sphere $x^2+y^2+z^2 = 4$.

$$\iint_S F_1\,dy\,dz + F_2\,dz\,dx + F_3\,dx\,dy = \iiint 2\,dx\,dy\,dz$$

$$= 2\int_V dv$$

$= 2$ Volume of the sphere

$$= 2 \cdot \frac{4}{3}\pi r^3$$

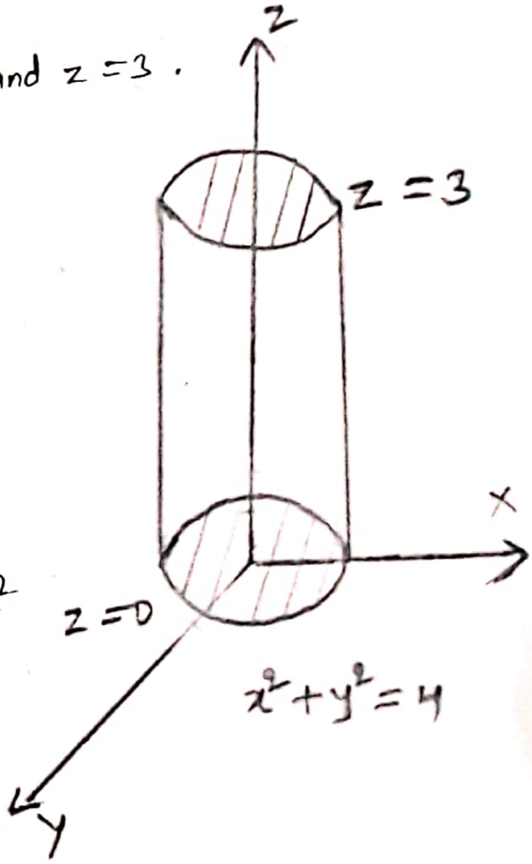
$$= 2 \cdot \frac{4}{3}\pi (2)^3 \quad [\because \text{Radius of the sphere } r = 2]$$

$$= \frac{64\pi}{3}$$

→ Verify divergence theorem for $\bar{F} = 4xi - 2y^2 j + z^2 k$ taken over the region bounded by $x^2+y^2=4$, $z=0$ and $z=3$.

Sol:- Given that $\bar{F} = 4xi - 2y^2 j + z^2 k$

The region bounded by $x^2+y^2=4$, $z=0$ and $z=3$.



WKT Gauss Divergence Theorem

$$\int_V \operatorname{div}\bar{F}\, dv = \iint_S \bar{F}\cdot\bar{n}\, ds.$$

To evaluate $\int_V \operatorname{div}\bar{F}\, dv$:-

WKT $\operatorname{div}\bar{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$

We have $\bar{F} = 4xi - 2y^2 j + z^2 k$

$F_1 = 4x \quad F_2 = -2y^2 \quad F_3 = z^2$

$\frac{\partial F_1}{\partial x} = 4 \quad \frac{\partial F_2}{\partial y} = -4y \quad \frac{\partial F_3}{\partial z} = 2z$

$\operatorname{div}\bar{F} = 4 - 4y + 2z$.

We have $z=0$ and $z=3$.

Given that $x^2+y^2=4$ —(1) $\Rightarrow y^2 = 4-x^2$

$y = \pm\sqrt{4-x^2}$

Put $y=0$ in (1), $x^2=4 \Rightarrow x=\pm 2$.

$$\therefore \int_V \operatorname{div}\bar{F}\,dv = \int_{x=-2}^{x=2}\int_{y=-\sqrt{4-x^2}}^{y=\sqrt{4-x^2}}\left[\int_{z=0}^{z=3}(4-4y+2z)\,dz\right]dy\,dx.$$

$$= \int_{x=-2}^{x=2}\int_{y=-\sqrt{4-x^2}}^{y=\sqrt{4-x^2}}\left[4z-4yz+z^2\right]_{z=0}^{z=3} dy\,dx.$$

$$= \int_{x=-2}^{x=2}\int_{y=-\sqrt{4-x^2}}^{y=\sqrt{4-x^2}}\left[(21-12y)\right]dy\cdot dx.$$

$$= \int_{x=-2}^{x=2}\left[21y-6y^2\right]_{y=-\sqrt{4-x^2}}^{y=\sqrt{4-x^2}} dx.$$

$$= \int_{x=-2}^{x=2} \left[21\sqrt{4-x^2} - 6(4-x^2) - \left(-21\sqrt{4-x^2} - 6(4-x^2)\right)\right] dx$$

$$= \int_{x=-2}^{x=2} 42\sqrt{4-x^2}\, dx.$$

Put $x = 2\cos\theta$

$dx = -2\sin\theta\, d\theta$

when $x = -2$, $\theta = \pi$

when $x = 2$, $\theta = 0$.

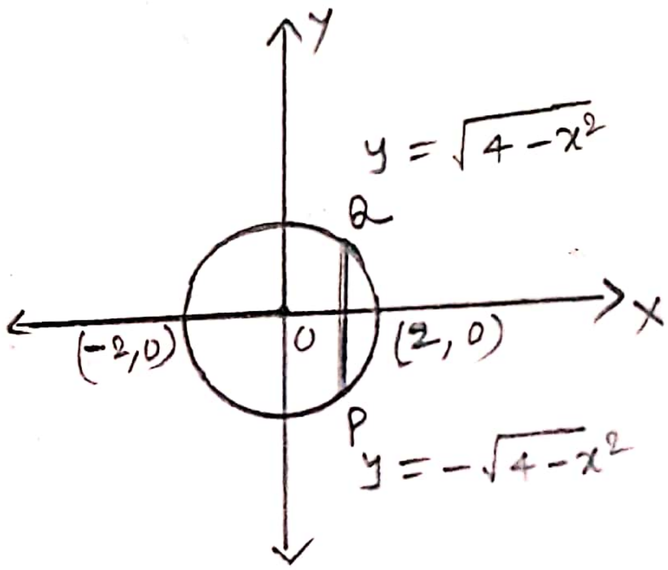
$$= 42 \int_{\theta=\pi}^{\theta=0} \sqrt{4-4\cos^2\theta}\,(-2\sin\theta)\, d\theta$$

$$= -84 \int_{\theta=\pi}^{\theta=0} 2\sin^2\theta\, d\theta$$

$$= 168 \int_{\theta=0}^{\theta=\pi} \left(\frac{1-\cos 2\theta}{2}\right) d\theta$$

$$= 84 \left[\theta - \frac{\sin 2\theta}{2}\right]_{\theta=0}^{\theta=\pi}$$

$$= 84 \left[\left(\pi - \frac{\sin 2\pi}{2}\right) - 0\right]$$



$$\int_V \operatorname{div}\bar{F}\, dv = 84\pi$$

To evaluate $\int_S \bar{F}\cdot\bar{n}\, ds$:-

The given surface of the cylinder can be divided into 3 parts.

(i) S_1 : the circular surface $z = 0$.

(ii) S_2 : the surface $z = 3$ (circular) and

(iii) S_3 : the cylindrical portion of $S : x^2+y^2=4,\ z=0,\ z=3$.

We now find $\iint \bar{F}\cdot\bar{n}\, ds$ over S_1, S_2, S_3.

$$\iint_S \bar{F}\cdot\bar{n}\, ds = \iint_{S_1} \bar{F}\cdot\bar{n}\, ds + \iint_{S_2} \bar{F}\cdot\bar{n}\, ds + \iint_{S_3} \bar{F}\cdot\bar{n}\, ds \quad \text{--- (2)}.$$

Case (i) :- To evaluate $\iint_{S_1} \bar{F}\cdot\bar{n}\, ds$. (OR) The circular surface $z=0$.

Unit normal vector to surface S_1, $z=0$ is $\bar{n}=-\bar{k}$

$$F = 4xi - 2y^2 j + z^2 k .$$

$$\bar{F} = 4xi - 2y^2 j \quad [\because z=0]$$

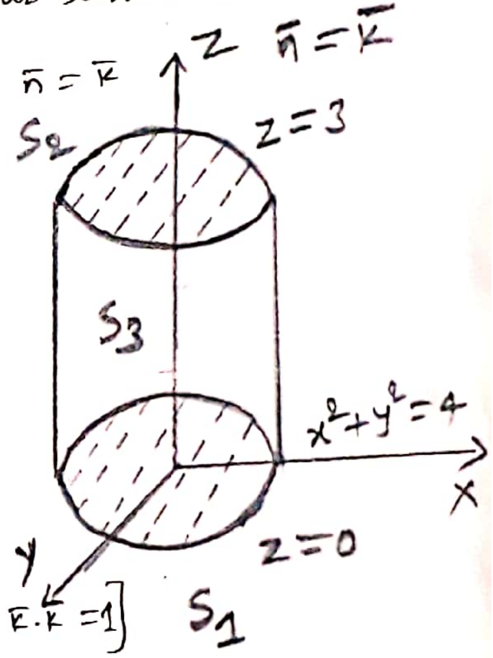
$$\bar{F}\cdot\bar{n} = [4xi - 2y^2 j]\cdot(-\bar{k})$$

$$\bar{F}\cdot\bar{n} = 0 .$$

$$\therefore \iint_{S_1} \bar{F}\cdot\bar{n}\, ds = 0 \quad \text{—— (3)}.$$

Case (ii) :- To evaluate $\iint_{S_2} \bar{F}\cdot\bar{n}\, ds$ (OR) The circular surface $z=3$.

Unit normal vector to surface S_2, $z=3$ is $\bar{n}=\bar{k}$



$$\bar{F} = 4xi - 2y^2 j + z^2 k$$

$$\bar{F} = 4xi - 2y^2 j + 9k \quad [\because z=3]$$

$$\bar{F}\cdot\bar{n} = [4xi - 2y^2 j + 9k]\cdot k$$

$$\bar{F}\cdot\bar{n} = 9.$$

$$\iint_{S_2} \bar{F}\cdot\bar{n}\, ds = \iint_R \bar{F}\cdot\bar{n}\, \frac{dx\,dy}{|\bar{n}\cdot\bar{k}|}$$

$$= \iint_R 9 .\ \frac{dx\,dy}{1} \quad [\because \bar{n}\cdot\bar{k} = \bar{k}\cdot\bar{k} = 1]$$

$$= 9 \iint_R dx\,dy$$

$$= 9\,[\text{Area of the circle } x^2+y^2=4]$$

$$= 9.\ \pi(2)^2$$

$$\iint_{S_2} \bar{F}\cdot\bar{n}\, ds = 36\pi \quad \text{—— (4)}$$

Case (iii) To evaluate $\iint_{S_3} \bar{F}\cdot\bar{n}\, ds$ (OR) The cylindrical portion of S: $x^2+y^2=4$, $z=0, z=3$.

Let $\phi = x^2+y^2-4$.

$$\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$

$$\nabla\phi = 2(xi+yj)$$

$$|\nabla\phi| = 2\sqrt{x^2+y^2}$$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2(xi+yj)}{2\sqrt{x^2+y^2}}$$

$$\hat{n} = \frac{xi+yj}{2} \quad [\because x^2+y^2=4]$$

$$\bar{F}\cdot\bar{n} = [4xi - 2y^2 j + z^2 k]\cdot\left(\frac{xi+yj}{2}\right)$$

$$= \frac{4x^2-2y^3}{2}$$

$$\bar{F}\cdot\bar{n} = 2x^2 - y^3.$$

To evaluate $\iint_{S_3} \bar{F}\cdot\bar{n}\,ds = \iint_{S_3} (2x^2-y^3)\,ds.$

Put $x = 2\cos\theta \quad y = 2\sin\theta$

$$ds = 2\,d\theta\,dz$$

Here z varies from 0 to 3

$\therefore$ z limits are $z=0$ $z=3$

θ limits are $\theta=0, \theta=2\pi$.

$$\iint_{S_3} \bar{F}\cdot\bar{n}\,ds = \int_{\theta=0}^{\theta=2\pi}\int_{z=0}^{z=3} (8\cos^2\theta - 8\sin^3\theta)\,2\,d\theta\,dz$$

$$= 16\int_{\theta=0}^{\theta=2\pi} (\cos^2\theta - \sin^3\theta)[z]_{z=0}^{z=3}\,d\theta.$$

$$= 48\int_{\theta=0}^{\theta=2\pi} [\cos^2\theta - \sin^3\theta]\,d\theta$$

$$= 48\int_{\theta=0}^{\theta=2\pi} \left[\frac{1+\cos 2\theta}{2} + \frac{\sin 3\theta}{4} - \frac{3\sin\theta}{4}\right] d\theta$$

$$= 48\left[\frac{\theta}{2} + \frac{\sin 2\theta}{4} - \frac{\cos 3\theta}{12} + \frac{3\cos\theta}{4}\right]_{\theta=0}^{\theta=2\pi}$$

$$= 48\pi \quad \text{——— (5)}$$

sub (3) (4) and (5) in (2), we get $\iint_S \bar{F}\cdot\bar{n}\,ds = 0 + 36\pi + 48\pi = 84\pi$

$$\therefore \int_V \operatorname{div}\bar{F}\,dv = \iint_S \bar{F}\cdot\bar{n}\,ds$$

$\therefore$ Gauss Divergence theorem verified.

→ Evaluate $\int_S (yz\,i + xz\,j + xy\,k) \cdot ds$ where S is the surface of the sphere $x^2+y^2+z^2=a^2$ in the first octant.

Sol:- Given that $\int_S (yz\,i + xz\,j + xy\,k) \cdot ds$. where S is the surface of the sphere. $x^2+y^2+z^2=a^2$ in the first octant.

The surface of the region V: OABC is piecewise smooth comprised of four surfaces.

(i) S_1 : circular quadrant OBC in the yz-plane.

(ii) S_2 : circular quadrant OCA in the zx-plane.

(iii) S_3 : circlcular quadrant OAB in the xy-plane.

(iv) S : Surface ABC of the sphere in the first octant.

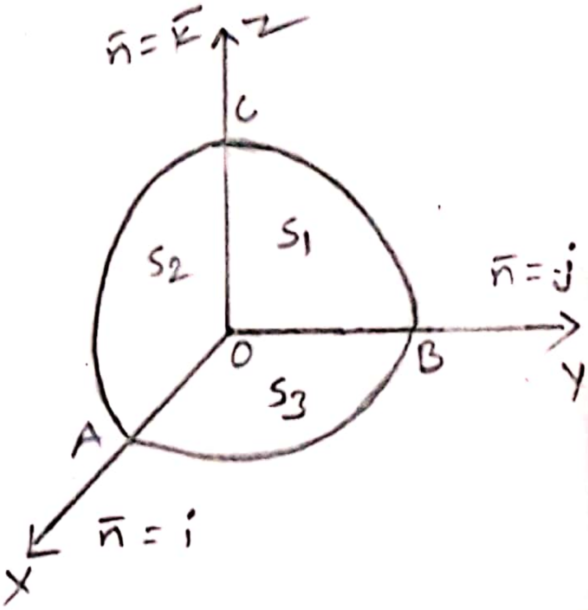
Let $\bar{F} = yz\,i + xz\,j + xy\,k$.

WKT Gauss Divergence theorem.

$$\int_V \operatorname{div}\bar{F}\, dv = \int_S \bar{F} \cdot \bar{n}\, ds \quad \text{—— (1)}$$

We can write $\int_V \operatorname{div}\bar{F}\, dv = \int_{S_1} \bar{F}\cdot\bar{n}\, ds + \int_{S_2} \bar{F}\cdot\bar{n}\, ds + \int_{S_3} \bar{F}\cdot\bar{n}\, ds + \int_S \bar{F}\cdot\bar{n}\, ds$.

We have $\operatorname{div}\bar{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$



$\bar{F} = yz\,i + zx\,j + xy\,k$

$F_1 = yz \quad F_2 = zx \quad F_3 = xy$

$\frac{\partial F_1}{\partial x} = \frac{\partial}{\partial x}(yz) = 0, \quad \frac{\partial F_2}{\partial y} = \frac{\partial}{\partial y}(zx) = 0$

$\frac{\partial F_3}{\partial z} = \frac{\partial}{\partial z}(xy) = 0$

$\operatorname{div}\bar{F} = 0.$

$\int_V \operatorname{div}\bar{F}\, dv = 0$ —— (2).

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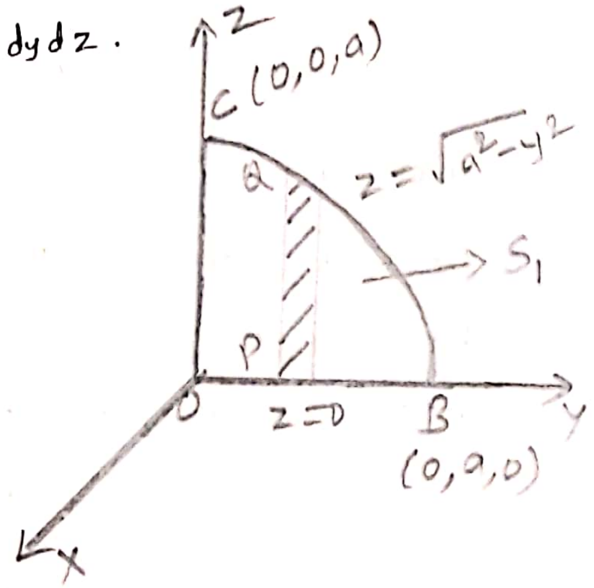
Case (ii):- To evaluate $\int_{S_1} \bar{F}\cdot\bar{n}\, ds$ (OR) on the surface S_1 :-

unit normal vector to the surface S_1 is $\bar{n} = -i$

The surface S_1 is in yz plane., $ds = dy\,dz$.



$$\int_{S_1} \bar{F}\cdot\bar{n}\, ds = \iint_R \bar{F}\cdot\bar{n}\,\frac{dy\,dz}{|n\cdot i|}$$

$$\bar{F}\cdot\bar{n} = [yz\,i + zx\,j + xy\,k]\cdot(-i)$$

$$\bar{F}\cdot\bar{n} = -yz$$

$$n\cdot i = -i\cdot i = -1$$

$$|n\cdot i| = 1.$$

Here y varies from o to a.

$\therefore$ y limits $y=0,\ y=a$.

For each y, z varies from $z=0$ to $\sqrt{a^2-y^2}$

$\therefore$ z limits $z=0,\ z=\sqrt{a^2-y^2}$

$$\int_{S_1} \bar{F}\cdot\bar{n}\, ds = \iint_R -yz\cdot\frac{dy\,dz}{1}$$

$$= -\int_{y=0}^{y=a}\int_{z=0}^{z=\sqrt{a^2-y^2}} yz\,dz\,dy$$

$$= -\int_{y=0}^{y=a} y\cdot\left[\frac{z^2}{2}\right]_{z=0}^{z=\sqrt{a^2-y^2}} dy$$

$$= -\frac{1}{2}\int_{y=0}^{y=a} y(a^2-y^2)\,dy$$

$$= -\frac{1}{2}\left[\frac{a^2y^2}{2} - \frac{y^4}{4}\right]_{y=0}^{y=a}$$

$$= -\frac{1}{2}\left[\left(\frac{a^4}{2} - \frac{a^4}{4}\right) - 0\right]$$

$$\int_{S_1} \bar{F}\cdot\bar{n}\, ds = -\frac{a^4}{8} \quad \text{---} \quad (3)$$

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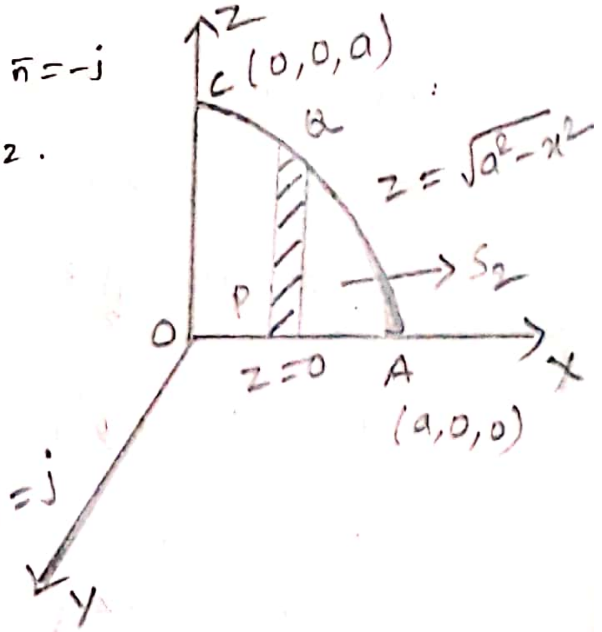
Case (ii) To evaluate $\int_{S_2} \bar{F}\cdot\bar{n}\,ds$ (OR) on the surface S_2 :-



Unitnormal vector to the surface S_2 is $\bar{n}=-j$

The surface S_2 is in xz plane $ds = dx\,dz$.

$$\int_{S_2} \bar{F}\cdot\bar{n}\,ds = \iint_R \bar{F}\cdot\bar{n}\,\frac{dx\,dz}{|n\cdot j|}$$

$$\bar{F}\cdot\bar{n} = [yz\,i + zx\,j + xy\,k]\cdot(-j)$$

$$\bar{F}\cdot\bar{n} = -zx.$$

$$n\cdot j = -j\cdot j = -1$$

$$|n\cdot j| = 1$$

Here x varies from 0 to a.

$\therefore$ x limits $x=0,\ x=a$.

For each x, z varies from 0 to $\sqrt{a^2-x^2}$.

$\therefore$ z limits $z=0,\ z=\sqrt{a^2-x^2}$

$$\int_{S_2} \bar{F}\cdot\bar{n}\,ds = \iint_R -zx\,\frac{dx\,dz}{1}$$

$$= -\int_{x=0}^{x=a}\int_{z=0}^{z=\sqrt{a^2-x^2}} xz\,dz\,dx$$

$$= -\int_{x=0}^{x=a} x\cdot\left[\frac{z^2}{2}\right]_{z=0}^{z=\sqrt{a^2-x^2}} dx$$

$$= \frac{-1}{2}\int_{x=0}^{x=a} x(a^2-x^2)\,dx.$$

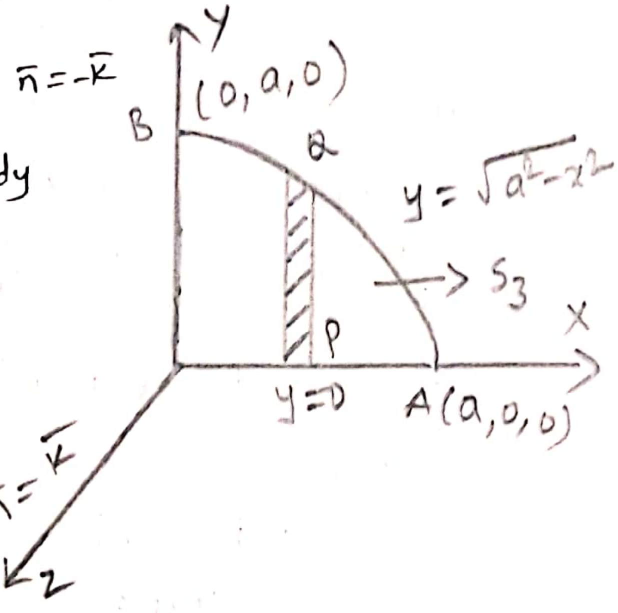
$$= \frac{-1}{2}\left[\frac{a^2x^2}{2} - \frac{x^4}{4}\right]_{x=0}^{x=a}$$

$$= \frac{-1}{2}\left[\left(\frac{a^4}{2} - \frac{a^4}{4}\right) - 0\right]$$

$$\int_{S_2} \bar{F}\cdot\bar{n}\,ds = -\frac{a^4}{8} \quad \text{------} \quad (4)$$

Case (iii) To evaluate $\int_{S_3} \bar{F}\cdot\bar{n}\, ds$ (OR) On the surface S_3 :-

Unit normal vector to the surface S_3 is $\bar{n} = -\bar{k}$



The surface S_3 is in xy plane, $ds = dx\,dy$

$$\int_{S_3} \bar{F}\cdot\bar{n}\, ds = \iint_R \bar{F}\cdot\bar{n}\,\frac{dx\,dy}{|\bar{n}\cdot k|}$$

$$\bar{F}\cdot\bar{n} = [yz\,i + xz\,j + xy\,k]\cdot(-\bar{k})$$

$$\bar{F}\cdot\bar{n} = -xy$$

$$\bar{n}\cdot k = -k\cdot k = -1$$

$$|\bar{n}\cdot k| = 1$$

Here x varies from 0 to a

$\therefore$ x limits $x=0,\ x=a$.

For each x, y varies from $y=0$ to $\sqrt{a^2-x^2}$

$\therefore$ y limits $y=0,\ y=\sqrt{a^2-x^2}$

$$\int_{S_3} \bar{F}\cdot\bar{n}\, ds = \iint_R -xy\,.\,\frac{dx\,dy}{1}$$

$$= -\int_{x=0}^{x=a}\int_{y=0}^{y=\sqrt{a^2-x^2}} xy\, dy\, dx$$

$$= -\int_{x=0}^{x=a} x\left[\frac{y^2}{2}\right]_{y=0}^{y=\sqrt{a^2-x^2}} dx$$

$$= -\frac{1}{2}\int_{x=0}^{x=a} x(a^2-x^2)\, dx$$

$$= -\frac{1}{2}\left[\frac{a^2x^2}{2} - \frac{x^4}{4}\right]_{x=0}^{x=a}$$

$$= -\frac{1}{2}\left[\left(\frac{a^4}{2}-\frac{a^4}{4}\right)-0\right]$$

$$\int_{S_3} \bar{F}\cdot\bar{n}\, ds = -\frac{a^4}{8} \quad \text{——} \quad (5)$$

sub. ② ③ ④ and ⑤ in ①, we get

$$0 = -\frac{a^4}{8} - \frac{a^4}{8} - \frac{a^4}{8} + \int_S \bar{F}.\bar{n}\,ds.$$

$$\therefore \int_S \bar{F}.\bar{n}\,ds = \frac{3a^4}{8}.$$

→ Use Divergence theorem to evaluate $\iint (xi+yj+z^2k)\cdot\bar{n}\,ds$ where S is the surface bounded by the cone $x^2+y^2=z^2$ in the plane $z=4$.

Sol: Given that $\iint (xi+yj+z^2k)\cdot\bar{n}\,ds$ where S is the surface bounded by the cone $x^2+y^2=z^2$ in the plane $z=4$.

Let $\bar{F} = xi+yj+z^2k$, $F = F_1 i + F_2 j + F_3 k$.

WKT Gauss Divergence theorem, we have

$$\iint_S \bar{F}\cdot\bar{n}\,ds = \int_V \operatorname{div}\bar{F}\,dv$$

$$\operatorname{div}\bar{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$\operatorname{div}\bar{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z^2).$$

$$\operatorname{div}\bar{F} = 2(1+z)$$

On the cone, $x^2+y^2=z^2$ and $z=4 \Rightarrow x^2+y^2=16$.

The limits are $z=0$ to 4, $y=0$ to $\sqrt{16-x^2}$, $x=0$, to $x=4$.

$$\int_S \bar{F}\cdot\bar{n}\,ds = \int_V \operatorname{div}\bar{F}\,dv = \int_{x=0}^{x=4}\int_{y=0}^{y=\sqrt{16-x^2}}\int_{z=0}^{z=4} 2(1+z)\,dz\,dy\,dx$$

$$= \int_{x=0}^{x=4}\int_{y=0}^{y=\sqrt{16-x^2}} 2\left[z+\frac{z^2}{2}\right]_{z=0}^{z=4} dy\,dx.$$

$$= 24\int_{x=0}^{x=4}\int_{y=0}^{y=\sqrt{16-x^2}} dy\,dx = 24\int_{x=0}^{x=4}\left[y\right]_{y=0}^{y=\sqrt{16-x^2}} dx.$$

$$= 24\int_{x=0}^{x=4}\sqrt{16-x^2}\,dx.$$

$$= 24\left[\frac{x}{2}\sqrt{16-x^2} + \frac{16}{2}\sin^{-1}\left(\frac{x}{4}\right)\right]_{x=0}^{x=4}$$

$$= 24\left[0+8\sin^{-1}(1)-0-0\right]$$

$$\iint_S \bar{F}\cdot\bar{n}\,ds = 96\pi$$

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1. Solution of Algebraic and Transcendental Equations

Algebraic Equation :—

An equation is of the form $f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \dots + a_{n-1} x + a_n = 0$ is called an algebraic equation of n^{th} degree where $a_0, a_1, a_2, \dots a_n$ are real numbers, n is a non -ve integer and $a_0 \neq 0$.

Eg:- $5x^2 - 7x + 8 = 0$, $4x^3 - 8x + 9 = 0$ are algebraic equation of degree 2, 3.

Transcendental Equation :—

If $f(x)$ contains the some other functions namely trigonometric, logarithmic, exponential etc then the eqn $f(x) = 0$ is called transcendental equation.

Eg $xe^x = \sin x$, $3x - \log_{10} x = 8$.

Zero (or) Root of a Equation :—

A number α (real or complex) is called a root (or solution) of the equation $f(x) = 0$ if $f(\alpha) = 0$. We can also say that α is a zero of the function $f(x)$.

Geometrically the root of $f(x) = 0$ is the value of x at which the graph of $f(x)$ meet the x-axis

A Polynomial Equation of degree n will have exactly n roots real, or complex, simple or multiple.

A number α is a simple root of $f(x) = 0$ if $f(\alpha) = 0$ and $f'(\alpha) \neq 0$

Then we can write $f(x) = (x - \alpha) g(x)$, $g(\alpha) \neq 0$.

A number α is a multiple root of multiplicity m of $f(x)=0$ if $f(\alpha)=f'(\alpha)=f''(\alpha)=\cdots=f^{(m-1)}(\alpha)=0$ and $f^{(m)}(\alpha)\neq 0$ Then $f(x)$ can be written as $f(x)=(x-\alpha)^m g(x)$, $g(\alpha)\neq 0$. (2)

Exact and Approximate Numbers :—

Exact Numbers are those which have definite value.

Eg:- $2, 7, 9, \frac{1}{2}, \frac{1}{15}, \pi, e$ etc.

Approximate numbers are those that represent that exact numbers to a certain degree of accuracy

Eg:- $\sqrt{2}\simeq 1.414$ $e\simeq 2.7183$, $\pi\simeq 3.142\ldots$ are not exact. since they contain infinitely many digits. The numbers obtained by retaining a few digits are called Approximate numbers.

Common Method of Rounding :—

This method is commonly used in accounting. This method is also known as Symmetric arithmetic rounding or round half up (Symmetric implementation).

→ Decide which is the last digit to keep.

→ Increase it by 1 if the next digit 5 or more (rounding up).

→ Leave the same if the next digit is 4 or less (rounding down).

Eg:- (i) 3.044 rounded to hundredths is 3.04 because the next digit is 4, which is less than 5.

(ii) 3.046 rounded to hundredths is 3.05 because the next digit is 6 which is more than 5.

(iii) 3.0447 rounded to hundredths is 3.04 because the next digit 4, is less than 5.

Significant figures :-

All the digits 1, 2, 3 ...... 9 are significant figures, '0' may or may not be significant figure. It depends on the context in which zero has been used

Zeroes concept :-

(i) Zeroes appearing between non zero digits are significant

Eg:- 60.8 has three significant figures.

3900 8 has five significant figures.

(ii) Zeroes appearing in front of non zero digits are not significant

Eg:- 0.093827 have 5 significant figures

0.0008 has one significant figure

(iii) Zeroes at the end of a number and to the right of a decimal are significant.

Eg:- 35.00 has four significant figures.

8,000.000000 has ten significant figures.

(iv) Zeroes at the end of a number without a decimal point may or may not be significant and are therefore ambiguous.

Eg:- 1,000 could have between one and four significant figures

This ambiguity could be resolved by placing a decimal after the number, e.g writing "1,000." to indicate specifically that four significant figures are meant, but this is a non-standard usage. To specify unambiguously how many significant figures are implied, scientific notation can be employed.

→ 1×10^3 has one significant figure, while

→ 1.000×10^3 has four.

Rules for counting Significant Digits :—

(1) Always count non zero digits.

Eg :- 21 has two significant figures while 8.926 has four.

(2) Never count leading zeroes.

Eg :- 021 and 0.021 both have two significant figures.

(3) Always count zeroes which fall somewhere between two non zero digits.

Eg :- 20.8 has 3 significant figures while 0.00104 0009 has seven.

(4) Count trailing zeros iff the number contains a decimal point.

Eg :- 210 and 210000 both have 2 significant figures while 210. has three 210.00 has five.

(5) For numbers expressed in scientific notation, ignore the exponent apply Rules 1-4 to the mantissa.

Eg :- 4.2010×10^{28} has five significant figures.

Rounding off :—

cutting off of some of the end digits of a number and retaining a fixed number of digits in numerical computation is called rounding off.

Rules of Rounding off numbers :—

(1) In rounding off numbers, the last figure kept should be unchanged if the first figure dropped is less than 5.

Eg :- If only one decimal is to be kept then 6.422 becomes 6.4.

(2) In rounding off numbers the last figure kept should be increased by 1 if the first figure dropped is greater than 5.

Bisection Method :—

(5)

This method is based on the repeated application of the intermediate value theorem to obtain an approximation to the root. Suppose that a root of $f(x)=0$ lies in the interval $I_0=(a_0, b_0)$, that is $f(a_0)f(b_0)<0$.

We bisect this interval and obtain $c_1=\frac{a_0+b_0}{2}$ Then, the root lies in the interval $I_1=(a_0, c_1)$ if $f(a_0)f(c_1)<0$.

otherwise, it lies in the interval $I_1=(c_1, b_0)$.

Thus the length of the interval I_1, is one half of that of I_0.

Continuing this procedure, we obtain a nested set of subintervals $I_0 \supset I_1 \supset I_2 \dots$ such that each of these subintervals contains the root.

After repeating the bisection procedure n times, we obtain an interval I_n of length $(b_0-a_0)/2^n$. which contains the root.

The mid-point of the last interval is taken as the required approximation to the root.

Note that the method does not use the value of $f(x)$, but only its sign. Hence if an accuracy for the root is prescribed, we can determine in advance for all equations, the number of iterations.

After the nth iteration, we have $\frac{b_0-a_0}{2^n} \leq \varepsilon$.

Taking the logarithms, we get

$$\log\left(\frac{b_0-a_0}{2^n}\right) \leq \log \varepsilon$$

$$\log(b_0-a_0) - \log 2^n \leq \log \varepsilon$$

$$\log(b_0-a_0) - n\log 2 \leq \log \varepsilon$$

$$n\log 2 \geq \log(b_0-a_0) - \log \varepsilon$$

$$\therefore n \geq \operatorname{Int}\left[\frac{\log(b_0-a_0)-\log \varepsilon}{\log 2}\right]$$

Where Int [...] stands for the next integer.

For example if $b_0 - a_0 = 1$ and $\varepsilon = 10^{-2}$ we get $n \geq 7$. (6)

that is seven iterations are required.

Note:- (i) The cost of the method is one function evaluation per iteration

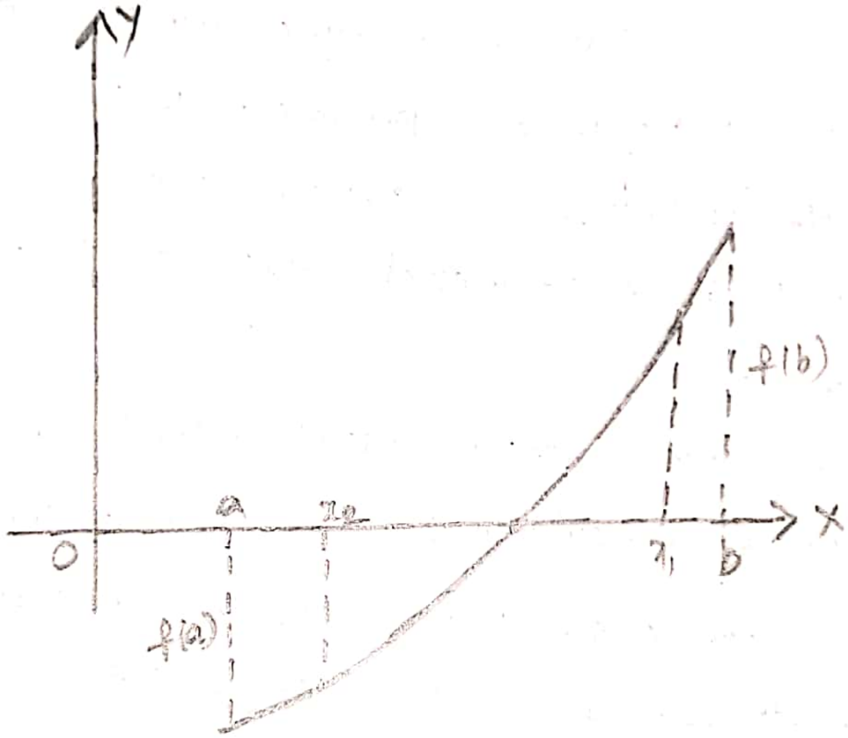
(ii) The method never fails as the root lies in the interval being considered

(iii) The value of $f(x_k)$ is not used, but only its sign.

(iv) At each iteration, the length of the interval is reduced by a factor of 2.

(v) The method has linear convergence.

(vi) The method is very slow, if high accuracy is required.



Eg:- If only two decimals are to be kept then 6.4872 becomes 6.49 similarly 6.997 becomes 7.00.

(3) In rounding off numbers if the first figure dropped is 5 and all the figures following the five are zero or if there are no figures after the 5, then the last figure kept should be unchanged if that last figure is even.

Eg: If only one decimal is to be kept then 6.65 becomes 6.6.

Eg: If only two decimals are to be kept then 7.485 becomes 7.48.

Rule (4) :- In rounding off numbers, if the first figure dropped is 5, and all the following the five are zero or if there are no figures after the 5, then the last figure kept should be increased by 1 if that last figure is odd.

Eg :- If only two decimals are to be kept then 6.755000 becomes 6.76.

Eg :- if only two decimals are to be kept 8.995 becomes 9.00.

(5) In rounding off numbers, if the first figure dropped is 5 and there are any figures following the five that are zero, then the last figure kept should be increased by 1.

Eg: If only one decimal is to be kept then 6.6501 becomes 6.7.

If only two decimals are to be kept then 7.4852007 becomes 7.49.

Number	Number of decimal places desired	Last figure to be kept	First figure to be dropped	Last figure kept and los number becomes
6.422	1	6.4	6.42	6.4
6.4872	2	6.48	6.487	6.49
6.997	2	6.99	6.997	7.00.
6.6500	1	6.6	6.65	6.6
7.485	2	7.48	7.485	7.48
6.755000	2	6.75	6.755	6.76
8.995	2	8 99	8.995	9.00.
6.6501	1	6.6	6.65	6.7
7.4852007	2	7.48	7.485	7.49.

Error :- If the difference b/n the Exact number and Approximate number are called the Error that means Exact is denoted by X, approximate is denoted by x. $E = X - x$.

Absolute Error :- Absolute Error is denoted by E_A and $E_A = |X - x|$ that means $E_A = |E|$.

Maximum Error is called Absolute Error.

Relative Error :- Relative Error is denoted by E_R and $E_R = \frac{E_A}{|X|}$

$$E_R = \frac{|X - x|}{|X|}$$

Percentage Error :- Percentage Error is denoted by E_p and

$E_p = E_R \times 100$. $\quad E_p = \frac{|X - x|}{|X|} \times 100$

The Methods of finding the root of $f(x)=0$ are classified as

(3)

(i) Direct Methods

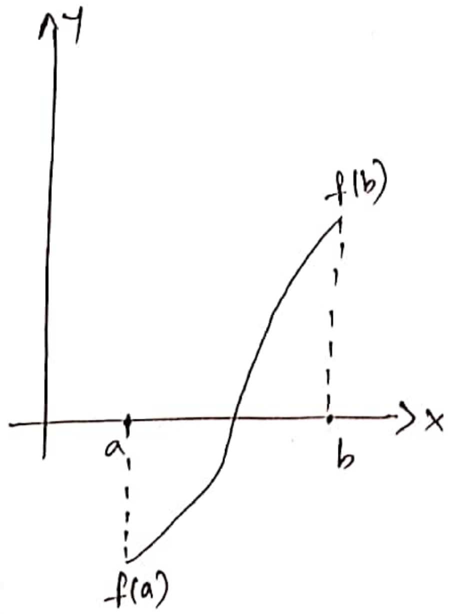
(ii) Numerical Methods.

Direct Methods give the exact values of all the roots in a finite number of steps.

Numerical Methods are based on the idea of successive approximations. In these methods we start with one or two initial approximations to the root and obtain a sequence of approximations $x_0, x_1, x_2, \ldots x_n$ which in the limit as $n \to \infty$ Converge to the exact root $x=a$.

There are no direct methods for solving higher degree algebraic Equations or transcendental Eqn's such eqns can be solved by Numerical methods.

<u>Theorem</u> :- Let the function $f(x)$ be continuous on $[a,b]$. Let $f(a) < 0$ and $f(b) > 0$ then $f(x)=0$ will have atleast one root between a and b.



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Bisection Method :-

Step 1 :- Consider the equation $f(x)=0$

Identity two points $x=a$ and $x=b$ such that $f(a)$ and $f(b)$ have opposite signs. Let $f(a)$ be -ve and $f(b)$ be +ve.

Than there will be a root of the equation $f(x)=0$ in b/n a and b.

Step 2 :- Find $x_1 = \frac{a+b}{2}$, calculate $f(x_1)$

If $f(x_1)=0$ then x_1 becomes the root of the equation $f(x)=0$.

otherwise,

(a) If $f(x_1)<0$ then the root lies between x_1 and b.

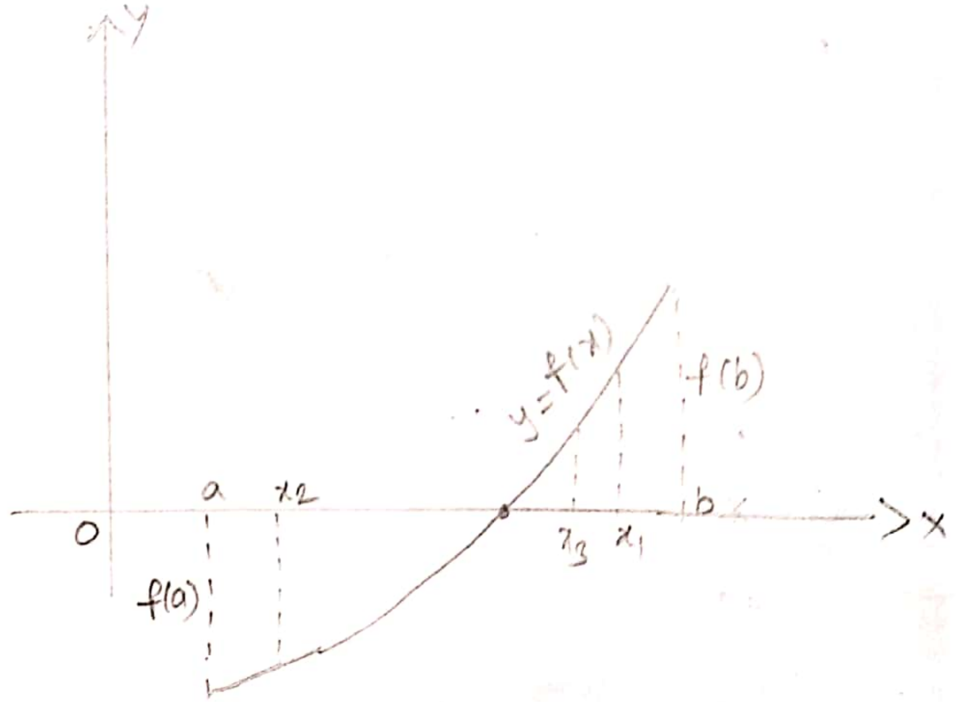
Find $x_2 = \frac{x_1+b}{2}$, calculate $f(x_2)$ and so on.

else

(b) If $f(x_1)>0$ then the root lies between a and x_1

Find $x_3 = \frac{a+x_1}{2}$, calculate $f(x_3)$ and so on.

We proceed in this way until the two successive approximations are approximately equal.



(1) Find a real root of the equation $x \log_{10} x = 1.2$ which lies between 2 and 3 by bisection method upto 5 approximation.

Sol:- Let $f(x) = x \log_{10} x - 1.2$

$x = 2, \quad f(2) = 2\log_{10} 2 - 1.2 = 0.60206 - 1.2 = -0.59794 < 0.$

$x = 3, \quad f(3) = 3\log_{10} 3 - 1.2 = 1.43136 - 1.2 = 0.23136 > 0.$

Since $f(2)$ and $f(3)$ are of opposite signs

$\therefore$ The given equation have root between 2 and 3.

First Approximation :-

$$x_1 = \frac{2+3}{2} = 2.5$$

$$f(x_1) = f(2.5) = 2.5\log_{10}(2.5) - 1.2 = 0.99485 - 1.2 = -0.2051 < 0.$$

We observe that $f(2.5)$ and $f(3)$ are of opposite signs.

$\therefore$ The root lies between 2.5 and 3.

Second Approximation :-

$$x_2 = \frac{2.5+3}{2} = 2.75$$

$$f(x_2) = f(2.75) = (2.75)\log_{10} 2.75 - 1.2 = 1.20816 - 1.2 = 0.00816 > 0$$

We observe that $f(2.5)$ and $f(2.75)$ are of opposite signs.

$\therefore$ The root lies between 2.5 and 2.75.

Third Approximation :-

$$x_3 = \frac{2.5+2.75}{2} = 2.625$$

$$f(x_3) = f(2.625) = (2.625)\log_{10} 2.625 - 1.2 = 1.1002 - 1.2 = -0.09978 < 0$$

We observe that $f(2.625)$ and $f(2.75)$ are of opposite signs.

$\therefore$ The root lies between 2.625 and 2.75

Fourth Approximation :—

$$x_4 = \frac{2.625 + 2.75}{2} = 2.6875$$

$$f(x_4) = f(2.6875) = 2.6875 \log_{10} 2.6875 - 1.2 = 1.153874 - 1.2 = -0.0461 < 0.$$

We observe that $f(2.6875)$ and $f(2.75)$ are of opposite signs.

$\therefore$ The root lies between 2.6875 and 2.75

Fifth Approximation :—

$$x_5 = \frac{2.6875 + 2.75}{2} = 2.71875$$

$$f(x_5) = f(2.71875) = (2.71875)\log_{10}(2.71875) - 1.2 = 1.18094 - 1.2$$
$$= -0.01906.$$

$\therefore$ An approximate root of the given equation $x\log_{10} x - 1.2$ is 2.71875.

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→ Find a positive root of the equation $x^3-4x-9=0$ using bisection method in five stages and correct to four decimal places. (4)

Sol: Let $f(x) = x^3-4x-9$.

$x=0$ $f(0) = 0-0-9 = -9 < 0$

$x=1$ $f(1) = 1-4-9 = -12 < 0$

$x=2$ $f(2) = 8-8-9 = -9 < 0$

$x=3$ $f(3) = 27-12-9 = 6 > 0$

We note that $f(2)$ and $f(3)$ are of opposite signs.

∴ The given equation $x^3-4x-9=0$ does have a real root between 2 and 3.

First Approximation :—

$$x_1 = \frac{2+3}{2} = 2.5$$

$f(x_1) = f(2.5) = (2.5)^3 - 4(2.5) - 9 = 15.625 - 10 - 9 = -3.375 < 0.$

We observe that $f(2.5)$ and $f(3)$ are of opposite signs.

∴ The root lies between 2.5 and 3.

Second Approximation :—

$$x_2 = \frac{2.5+3}{2} = 2.75$$

$f(x_2) = f(2.75) = (2.75)^3 - 4(2.75) - 9 = 20.796875 - 11 - 9 = 0.796875 > 0.$

We observe that $f(2.5)$ and $f(2.75)$ are of opposite signs.

∴ The root lies between 2.5 and 2.75.

Third Approximation :—

$$x_3 = \frac{2.5+2.75}{2} = 2.625$$

$f(x_3) = f(2.625) = (2.625)^3 - 4(2.625) - 9$

$= 18.08789063 - 10.5 - 9 = -1.41211 < 0.$

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We observe that $f(2.625)$ and $f(2.75)$ are of opposite signs.

$\therefore$ The root lies between 2.625 and 2.75

<u>Fourth Approximation</u> :-

$$x_4 = \frac{2.625+2.75}{2} = 2.6875$$

$$f(x_4) = f(2.6875) = (2.6875)^3 - 4(2.6875) - 9$$

$$= 19.41088867 - 10.75 - 9 = -0.33911 < 0$$

We observe that $f(2.6875)$ and $f(2.75)$ are of opposite signs.

$\therefore$ The root lies between 2.6875 and 2.75.

<u>Fifth Approximation</u> :-

$$x_5 = \frac{2.6875+2.75}{2} = 2.71875$$

This is an approximate value of the required root of the given equation, obtained at the fifth stage of bisection.

$\therefore$ An approximate root of given equation is $x^3-4x-9=0$ is 2.71875.

→ Using the bisection method find an approximate root of the equation $x - \cos x = 0$ that lies between 0.5 and 1 (Here x is in radians) carry out five steps of approximations.

Sol: Let $f(x) = x - \cos x$

$x = 0.5$ $\quad f(0.5) = 0.5 - \cos(0.5) = 0.5 - 0.87758 = -0.37758 < 0.$

$x = 1$ $\quad f(1) = 1 - \cos(1) = 1 - 0.540302 = 0.4597 > 0.$

We note that $f(0.5)$ and $f(1)$ are of opposite signs

∴ The given equation does have a real root between 0.5 and 1.

First Approximations :—

$$x_1 = \frac{0.5+1}{2} = 0.75$$

$f(x_1) = f(0.75) = 0.75 - \cos(0.75) = 0.75 - 0.73169 = 0.01831 > 0.$

We observe that $f(0.5)$ and $f(0.75)$ are of opposite signs.

∴ The root lies between 0.5 and 0.75.

Second Approximations :—

$$x_2 = \frac{0.5+0.75}{2} = 0.625$$

$f(x_2) = f(0.625) = 0.625 - \cos(0.625) = 0.625 - 0.81096 = -0.18596 < 0.$

We observe that $f(0.625)$ and $f(0.75)$ are of opposite signs.

∴ The root lies between 0.625 and 0.75

Third Approximations :-

$$x_3 = \frac{0.625+0.75}{2} = 0.6875$$

$f(x_3) = f(0.6875) = 0.6875 - \cos(0.6875) = 0.6875 - 0.7728 = -0.0853 < 0.$

We observe that $f(0.6875)$ and $f(0.75)$ are of opposite signs.

$\therefore$ The root lies between 0.6875 and 0.75.

Fourth Approximation :—

$$x_4 = \frac{0.6875+0.75}{2} = 0.71875$$

$f(x_4) = f(0.71875) = 0.71875 - \cos(0.71875) = 0.71875 - 0.75263 = -0.033 < 0.$

We observe that $f(0.71875)$ and $f(0.75)$ are of opposite signs.

$\therefore$ The root lies between 0.71875 and 0.75.

Fifth Approximation :-

$$x_5 = \frac{0.71875+0.75}{2} = 0.734375.$$

This is an approximate value of the required root of the given equation obtained at the fifth stage of bisection.

$\therefore$ An approximate root of given equation $x - \cos x = 0$ is 0.734375.

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$f(x_7) = f(1.11328125) = 1.11328125 \sin(1.11328125) - 1 = 0.99878 - 1$

$= -0.00216 < 0$

We observe that 1.11328125 and 1.1171875 are of opposite signs

Eighth Approximation :-

-ve +ve +ve

$f(1.11328125) < 0$ $f(1.115234375) > 0$ $f(1.1171875) > 0$

1.11328125 1.115234375 1.1171875

$$x_8 = \frac{1.11328125 + 1.1171875}{2} = 1.115234375$$

$$f(x_8) = f(1.115234375) = (1.115234375) \sin(1.115234375) - 1$$
$$= 1.00149 - 1 = 0.00149 > 0.$$

We observe that $f(1.11328125)$ and $f(1.115234375)$ are of opposite signs.

$\therefore$ The root lies between 1.11328125 and 1.115234375.

Ningth Approximation :-

-ve +ve +ve

$f(1.11328125) < 0$ $f(1.114257813) > 0$ $f(1.115234375) > 0$

1.11328125 1.114257813 1.115234375

$$x_9 = \frac{1.11328125 + 1.115234375}{2} = 1.114257813.$$

$f(x_9) = f(1.114257813) = 1.114257813 \sin(1.114257813) - 1 = 1.00014 - 1 = 0.0001 > 0.$

We observe that $f(1.11328125)$ and $f(1.114257813)$ are of opposite signs.

$\therefore$ The root lies between 1.11328125 and 1.114257813.

Tenth Approximation :-

-ve -ve +ve

$f(1.11328125) < 0$ $f(1.113769532) < 0$ $f(1.114257813) > 0$

1.11328125 1.113769532 1.114257813

$$x_{10} = \frac{1.11328125 + 1.114257813}{2} = 1.113769532$$

$f(x_{10}) = f(1.113769532) = 1.113769532 \sin(1.113769532) - 1 = 0.99946 - 1 = -0.0005 < 0.$

We observe that $f(1.113769532)$ and $f(1.114257813) > 0$ are of opposite signs.

$\therefore$ The root lies between 1.113769532 and 1.114257813.

Eleventh Approximation :-

$$x_{11} = \frac{1.1137695312 + 1.1142578125}{2} = 1.1140136718$$

$f(x_{11}) = f(1.1140136718) = 1.1140136718 \sin(1.1140136718) - 1 = 0.9998 - 1 = -0.000199 < 0$

We observe that 10^{th} and 11^{th} approximations are approximately equal. This is an approximate value of the required root of the given equation obtained at the 11^{th} stage of bisection.

$\therefore$ An approximate root of given equation $x \sin x - 1 = 0$ is 1.1140.

BISECTION METHOD

1 Explain Bisection method with geometrical interpretation.

2 Find an approximate value of the root of the equation $x^3 - x - 11 = 0$ that lies between 2 and 3 using bisection method.

3 Find by using the bisection method, an approximate root of the equation $x^4 - x^3 - 2x^2 - 6x - 4 = 0$. that lies in the interval $(2, 2.75)$.

4 Find the nagative root of the equation $x^3 - 4x + 9 = 0$ using bisection. method.

5 Find the nagative root of the equation $x^3 - 3x + 4 = 0$ using bisection. method.

6 Find an approximate root of the equation. $x + \log_e^x = 5$ in $(3.2, 4)$ using bisection method.

7 Find the root of the equation $x = e^{-x}$ in $(0,1)$ using bisection method.

8 Find the root of the equation $2x - \log_{10}^x = 7$ in $(3.5, 4)$ using bisection method.

9. Find the root of the equation $x^2 - \log x = 12$ in $(3,4)$ by bisection method.

10. Find the root of the equation $e^x \sin x = 1$ in $(2,3)$ in radians using bisection method.

11. Using bisection method, find an approximate root of the equation. $\log x - \cos x = 0$ in $(1,2)$ in radians.

12 Using bisection method, find the root of $\sin x - 2x + 1 = 0$.

13. Find the root of the equation $x^2 + x - \cos x = 0$ using bisection method.

14. Find an approximate value of square root of 15 using bisection method.

15 Find an approximate value of cube root of 9 using bisection method.

16. Find an approximate value of 5th root of 5 using bisection method.

17 Prove that the order of convergence of Bisection method is linear

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→ By using the bisection method, find an approximate root of the equation $\sin x = \frac{1}{x}$ that lies between $x=1$ and $x=1.5$ (measured in radians) and correct to four decimal places.

Sol:- Given that $\sin x = \frac{1}{x}$ i.e $x\sin x - 1 = 0$.

Let $f(x) = x\sin x - 1$.

$x=1$ $\quad f(1) = \sin 1 - 1 = 0.84147 - 1 = -0.1585 < 0$.

$x=1.5$ $\quad f(1.5) = 1.5\sin(1.5) - 1 = 1.4962 - 1 = 0.4962 > 0$.

We note that $f(1)$ and $f(1.5)$ are of opposite signs

∴ The given equation $x\sin x - 1 = 0$ has a root between 1 and 1.5.

First Approximation :-

$-ve$	$+ve$	$+ve$
$f(1)<0$	$f(1.25)>0$	$f(1.5)>0$
1	1.25	1.5

$$x_1 = \frac{1+1.5}{2} = 1.25$$

$f(x_1) = f(1.25) = 1.25\sin(1.25) - 1 = 1.18623 - 1 = 0.18623 > 0$.

We observe that $f(1)$ and $f(1.25)$ are of opposite signs.

∴ The root lies between 1 and 1.25.

Second Approximation :-

$-ve$	$+ve$	$+ve$
$f(1)<0$	$f(1.125)>0$	$f(1.25)>0$
1	1.125	1.25

$$x_2 = \frac{1+1.25}{2} = 1.125.$$

$f(x_2) = f(1.125) = 1.125\sin(1.125) - 1 = 1.01505 - 1 = 0.01505 > 0$.

We observe that $f(1)$ and $f(1.125)$ are of opposite signs.

∴ The root lies between 1 and 1.125.

Third Approximation :-

$-ve$	$-ve$	$+ve$
$f(1)<0$	$f(1.0625)<0$	$f(1.125)>0$
1.	1.0625	1.125

$$x_3 = \frac{1+1.125}{2} = \frac{2.125}{2} = 1.0625.$$

$f(x_3) = f(1.0625) = 1.0625\sin(1.0625) - 1 = 0.92817 - 1 = -0.077183 < 0$.

We observe that $f(1.0625)$ and $f(1.125)$ are of opposite signs.

$\therefore$ The root lies between 1.0625 and 1.125.

Fourth Approximation :—

-ve	-ve	+ve
$f(1.0625)<0$	$f(1.09375)<0$	$f(1.125)>0$
1.0625	1.09375	1.125

$$x_4 = \frac{1.0625+1.125}{2} = \frac{2.1875}{2} = 1.09375$$

$$f(x_4) = f(1.09375) = 1.09375\sin(1.09375)-1 = 0.9718-1 = -0.028 < 0$$

We observe that $f(1.09375)$ and $f(1.125)$ are of opposite signs.

$\therefore$ The root lies between 1.09375 and 1.125.

Fifth Approximation :—

-ve	-ve	+ve
$f(1.09375)<0$	$f(1.109375)<0$	$f(1.125)>0$
1.09375	1.109375	1.125

$$x_5 = \frac{1.09375+1.125}{2} = \frac{2.21875}{2} = 1.109375$$

$$f(x_5) = f(1.109375) = 1.109375\sin(1.109375)-1 = 0.9934-1 = -0.0066 < 0$$

We observe that $f(1.109375)$ and $f(1.125)$ are of opposite signs.

$\therefore$ The root lies between 1.109375 and 1.125.

Sixth Approximation :—

-ve	+ve	+ve
$f(1.109375)<0$	$f(1.1171875)>0$	$f(1.125)>0$
1.109375	1.1171875	1.125

$$x_6 = \frac{1.109375+1.125}{2} = \frac{2.234375}{2} = 1.1171875$$

$$f(x_6) = f(1.1171875) = 1.1171875\sin(1.1171875)-1 = 1.0042-1 = 0.0042 > 0$$

We observe that $f(1.1171875)$ and $f(1.109375)$ are of opposite signs.

$\therefore$ The root lies between 1.109375 and 1.1171875.

Seventh Approximation :—

-ve	-ve	
$f(1.109375)<0$	$f(1.11328125)<0$	$f(1.1171875)>0$
1.109375	1.11328125	1.1171875

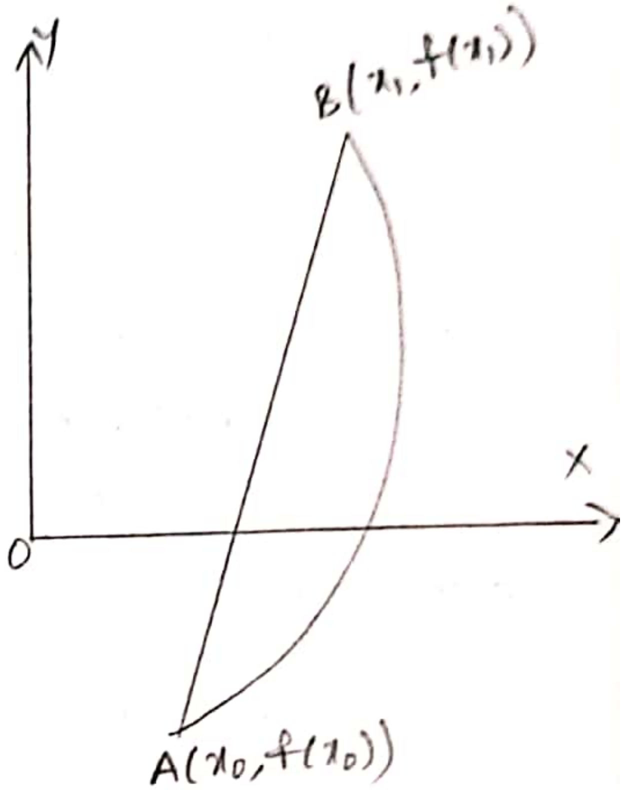
$$x_7 = \frac{1.109375+1.1171875}{2} = \frac{2.2265625}{2} = 1.11328125$$

Regula Falsi Method Formula :—

Consider the equation $f(x)=0$

Let x_0 and x_1 be two values of x such that $f(x_0)$ and $f(x_1)$ are opposite signs. since the graph of $y=f(x)$ crosses x-axis, the root must lies between x_0 and x_1

The chord joining $A(x_0, f(x_0))$ and $B(x_1, f(x_1))$ meets x-axis



Equation of the line AB is

$$y - f(x_0) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0)$$

The point of intersection of the line with x-axis will be the first approximation of the root of $f(x)=0$.

Let it be $(x_2, 0)$. At this point $y=0$.

$$0 - f(x_0) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x_2 - x_0)$$

$$x_2 - x_0 = \frac{-f(x_0)(x_1 - x_0)}{f(x_1) - f(x_0)}$$

$$x_2 = x_0 - \frac{f(x_0)(x_1 - x_0)}{f(x_1) - f(x_0)}$$

$$= \frac{x_0 f(x_1) - x_0 f(x_0) - x_1 f(x_0) + x_0 f(x_0)}{f(x_1) - f(x_0)}$$

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

If $f(x_0)$ and $f(x_2)$ are of opposite signs then the root lies between x_0 and x_2. Put $x_1 = x_2$, then we get next approximation.

If $f(x_1)$ and $f(x_0)$ are of opposite signs then the root lies between x_1 and x_2

Put $x_0 = x_2$ we get next approximation.

We proceed in this way till the two successive approximations are approximately equal.

The Regula Falsi

$$x_{i+1} = \frac{x_{i-1} f(x_i) - x_i f(x_{i-1})}{f(x_i) - f(x_{i-1})}.$$

Working Procedure :-

Consider the equation $f(x) = 0$.

Step 1 :- Consider two initial approximations to the root are x_0, x_1.

Find $f(x_0)$, $f(x_1)$. Assume that $f(x_0) < 0$, $f(x_1) > 0$.

$\therefore$ The root lies between the points x_0, x_1.

Step 2 :- Find x_2 using the formula. $x_2 = \dfrac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$.

Calculate $f(x_2)$. If suppose $f(x_2) > 0$.

$\therefore$ The root lies between the points x_0, x_2.

Step 3 :- Replace x_0 by x_1.

Find x_3 using the formula $x_3 = \dfrac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)}$.

Calculate $f(x_3)$. If suppose $f(x_3) < 0$.

$\therefore$ The root lies between the points x_2, x_3.

Step 4 :- We proceed in this way until the two successive approximations are approximately equal.

Regula Falsi Method

Let us consider the equation $f(x)=0$

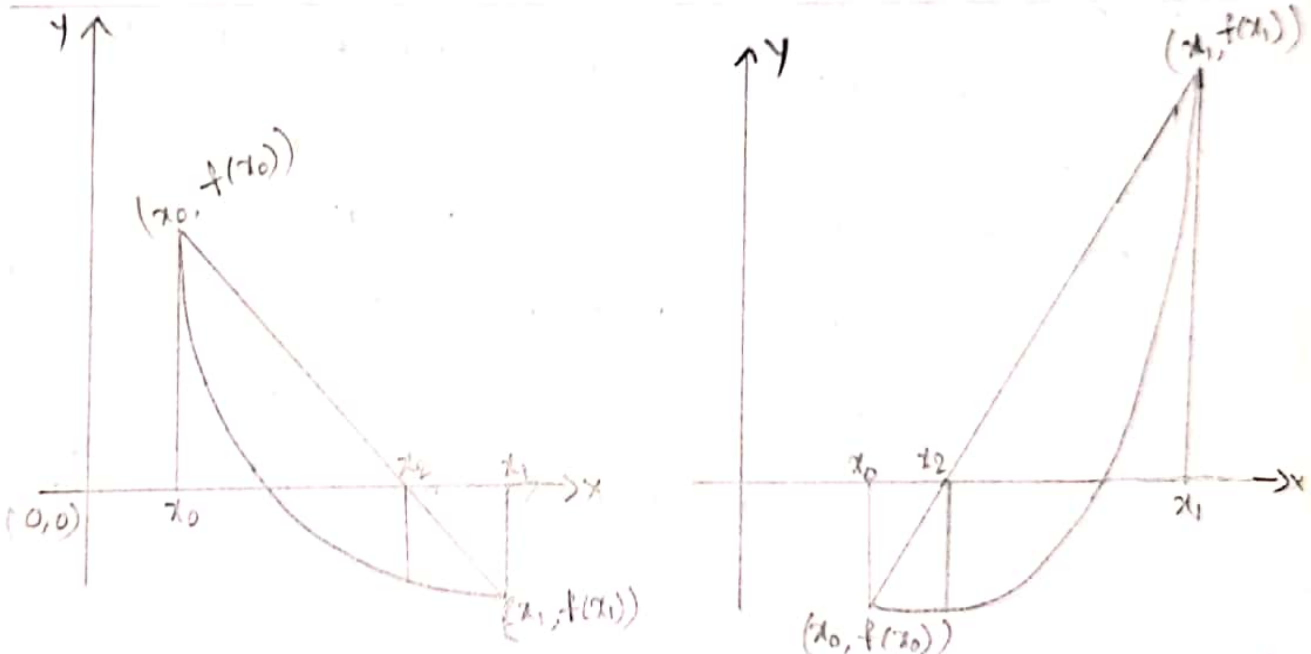
Consider two initial approximate values x_0 and x_1 near the required root so that $f(x_0)$ and $f(x_1)$ have different signs

This implies that a root lies between x_0 and x_1

The curve $f(x)$ crosses x-axis only once at the point x_2 lying between the points x_0 and x_1.

Consider the point $A(x_0, f(x_0))$ and $B(x_1, f(x_1))$ on the graph and suppose they are connected by a straight line.

Suppose this line cuts x-axis at x_2 We calculate the value of $f(x_2)$ at the point. If $f(x_0)$ and $f(x_2)$ are of opposite signs then the root lies between x_0 and x_2 and value x_1 is replaced by x_2 Otherwise the root lies between x_2 and x_1 and the value x_0 is replaced by x_2.



Another line is drawn by connecting the newly obtained pair of values. Again the point here the line cuts the x-axis is a closer approximation to the root. This process is repeated as many times as required to obtain the desired acuracy.

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Note :- (i) The method requires two initial approximations to the root

(ii) The method always converges to the root

(iii) The cost of the method is one evaluation of $f(x)$ per iteration.

(iv) If the root lies initially in (x_0, x_1), then one of the end points is fixed for all iterations. In figure 1., the end point x_0 is fixed while in figure 2, the end point x_1 is fixed. Then the method is of the form.
$$x_{i+1} = \frac{x_0 f_i - x_i f_0}{f_i - f_0} \quad i = 1, 2, 3, \ldots$$

This is the disadvantage of the method. However, it can be speeded up by inserting a bisection iteration after few iterations of the method of false position.

(v). The method has linear convergence.

Demerits of Regula Falsi Method :—

(i) Always one has to find the interval $[a,b]$ which brackets the zero of $f(x)=0$. more importantly $f(a) * f(b)$ must be less than zero.

(ii) Although the length of the interval is getting smaller in each iteration, it is possible that it may not go to zero. If the graph $y = f(x)$ is Concave near the root 'c'. one of the endpoints becomes fixed and the other one marches towards the root.

→ Find a real root for $e^x \sin x = 1$ using Regula Falsi method.

Sol. - Given that the equation is $e^x \sin x - 1 = 0$

Let $f(x) = e^x \sin x - 1$

The sin function values measured in radians

$x = 0.5$ $\quad f(0.5) = e^{0.5}\sin(0.5) - 1 = 0.7904390832 - 1 = -0.20956 < 0$

$x = 0.6$ $\quad f(0.6) = e^{0.6}\sin(0.6) - 1 = 1.028845866 - 1 = 0.02885 > 0$

$x = 0.7$ $\quad f(0.7) = e^{0.7}\sin(0.7) - 1 = 1.297295112 - 1 = 0.297295 > 0$

We note that $f(0.5)$ and $f(0.6)$ are of opposite signs

∴ The given equation does have a real root between 0.5 and 0.6.

By Regula Falsi method

$$x_{i+1} = \frac{x_{i-1} f(x_i) - x_i f(x_{i-1})}{f(x_i) - f(x_{i-1})}$$

<u>First</u> <u>Approximation</u> :-

$i = 1$. $\quad x_2 = \dfrac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$

Let $x_0 = 0.5$ $\quad f(x_0) = f(0.5) = -0.20956$

$x_1 = 0.6$ $\quad f(x_1) = f(0.6) = 0.02885$

$$x_2 = \frac{0.5(0.02885) - 0.6(-0.20956)}{0.02885 - (-0.20956)}$$

$$x_2 = \frac{0.014425 - 0.125736}{0.23841} = 0.5879$$

$f(x_2) = f(0.5879) = e^{0.5879}\sin(0.5879) - 1 = 0.99[illegible] - 1 = -0.00158 < 0$

We observe that $f(0.5879)$ and $f(0.6)$ are of opposite signs

∴ The root lies between 0.5879 and 0.6

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Second Approximation :-

$$i=2, \quad x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)}$$

$x_1 = 0.5879, \quad f(x_1) = f(0.5879) = -0.00158$

$x_2 = 0.6 \qquad f(x_2) = f(0.6) = 0.02885$

$$x_3 = \frac{0.5879(0.02885) - 0.6(-0.00158)}{0.02885 - (-0.00158)}$$

$$x_3 = \frac{0.016960915 + 0.000948}{0.03043} = 0.5885$$

$f(x_3) = f(0.5885) = e^{0.5885} \sin(0.5885) - 1 = -0.0000818 < 0.$

We observe that $f(0.5885)$ and $f(0.6)$ are of opposite signs.

$\therefore$ The root lies between 0.5885 and 0.6.

Third Approximation :—

$$i=3, \quad x_4 = \frac{x_2 f(x_3) - x_3 f(x_2)}{f(x_3) - f(x_2)}$$

$x_2 = 0.5885 \quad f(x_2) = f(0.5885) = -0.0000818$

$x_3 = 0.6 \qquad f(x_3) = f(0.6) = 0.02885$

$$x_4 = \frac{0.5885(0.02885) - 0.6(0.0000818)}{0.02885 - (-0.0000818)}$$

$$x_4 = \frac{0.016978225 + 0.00004908}{0.0289318} = 0.5885.$$

We observe that second and third approximations are approximately equal.

$\therefore$ An approximate root of the given equation is 0.5885.

mated values of a repeated root.

(ii) Most severe limitation in the use of this method is the requirement that $f'(x) \neq 0$ in the neighbourhood of the root α. Even a moderate value of $f'(x_0)$ may more than sampled by a large value of either $f(x_0)$ or $f'(x_0)$ to produce a value x that will result in a sequence that convergence to a root that we are not interested.

(iii) Since two function evaluations are required in each iteration Newton Raphson method requires more computing time.

Merits and demerits of Newton Raphson Method :-

Merits.

(i) In this method convergence is quite fast provided the starting value is close to the desired root.

(ii) If the root is simple, the convergence is quadratic.

(iii) The accuracy of Newtons method for the function $f(x)$ which possess continues first and second derivatives can be estimated. If $M = \max|f''(x)|$ and $m = \min|f''(x)|$ in an interval that contains the root α and the estimator x_1 and x_2 then.

$$|x_2 - \alpha| \leq (x-\alpha)^2 \frac{M}{m}$$

Thus the error decreases if $\left|(x_1-\alpha)^2 \frac{M}{m}\right| < 1$

(iv) Newton Raphson iteration is a single point iteration.

(v) This method can used for solving both algebraic and transcen-dental equations. It can also be used when the roots are complex

(vi) The convergence of Newton Raphson method is of quadratic convergence. This method converges more rapidly than other methods.

(vii) This is a powerful and elegent method to find the root of an equation. This method is used to improve the results obtained by the previous methods.

Demerits :-

(i) In deriving the formula for this method, it is assumed that α is a not a repeated root of $f(x)=0$. In this case the convergence of the iteration is not guaranteed. Thus the Newton Raphson method is not applicable to find the approxi-

Merits and demerits of Newton Raphson Method :-

Merits :-

1. In this method convergence is quite fast provided the starting value is close to the desired root.
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Demerits :-

1. In deriving the formula for this method, it is assumed that α is a not a repeated root of $f(x)=0$. In this case the convergence of the iteration is not guarented. Thus the Newton Raphson method is not applicable to find the approximated values of a repeated root.
2. Most severe limitation in the use of this method is the require-ment that $f'(x) \neq 0$ in the neighbourhood of the root α.

Even a moderate value of $f'(x_0)$ may more than sampled by a large value of either $f(x_0)$ or $f'(x_0)$ to produce a value x that will result In a sequence the converges to a root that we are not interested.

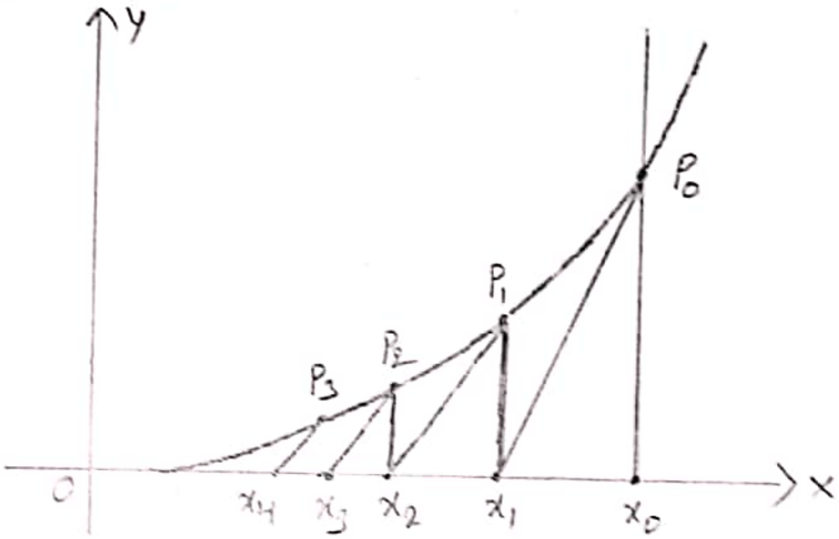
Geometrical significance :-

The Newton Raphson method starts with an initial approximation say x_0. Then a tangent is drawn from the corresponding point $f(x_0)$ on the curve $y = f(x)$

Let this tangent cuts the x-axis at a point say x_1.

Which will be a better approximation of the root.

Now calculate $f(x_1)$ and draw another tangent at the point $f(x_1)$ on the curve so that it cuts the x-axis at the point say x_2. The value of x_2 gives still better approximation and the process can be continued till the desired accuracy has been achieved.



Note:- (i) The method required one initial approximation.

(ii) The cost of the method is one evaluation of $f(x)$ and one evaluation of $f'(x)$ per iteration.

(iii) The method may fail if the initial approximation x_0 is far away from the root.

(iv) The method has second order convergence.

(v) This method fails if $f'(x) = 0$.

(vi) Newton's formula converges if $|f(x)f''(x)| < |f'(x)|^2$

Newton Raphson Method (Method of Tangents) :—

The Newton Raphson method is more advanced method in finding the root of the equation $f(x)=0$. It is used to improve the result obtained by Bisection or Regula falsi method.

Let $f(x)=0$ be the given equation.

Let x_0 be the approximate root of $f(x)=0$.

If x_1 is the exact root of the equation then $f(x_1)=0$

$$f(x_1) = f(x_0+(x_1-x_0)) = 0$$

By Taylor Series.

$$f(a+h) = f(a) + \frac{h}{1!}f'(a) + \frac{h^2}{2!}f''(a) + \cdots$$

$$f(x_1) = f(x_0) + (x_1-x_0)f'(x_0) + \frac{(x-x_0)^2}{2!}f''(x_0) + \cdots = 0.$$

Suppose $x_1 - x_0$ is very small then the higher powers can be neglected.

$$\therefore f(x_0) + (x_1-x_0)f'(x_0) = 0$$

$$(x_1-x_0)f'(x_0) = -f(x_0)$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Similarly $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$

Generally $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

This is the Newton Raphson formula

(10) Find the root of the equation $x^3 - 4x^2 + x - 10 = 0$ in $(4, 5)$ using Method of chords.

(11) Find an approximate root of $xe^x = 3$ using Method of false position.

12) Find the root of $x^3 - 9x + 1 = 0$ using Regula Falsi method.

13) Explain Regula Falsi Method with geometrical interpretation.

14 Prove that the order of convergence of Regula Falsi method is 1.

REGULA FALSI METHOD.

(1) By using the Method of False position, find an approximate root of the equation $x^3 - x - 10 = 0$ that lies between 1.8 and 2. Carry out three approximations. Ans: 1.8555.

(2) Using the method of false position, find the fifth root of 10 correct to four significant figures, assuming that the root lies between 1 and 2.

Ans:-

(3) By using the method of false position, find the root, correct to three decimal places of the equation $x \log_{10} x = 1.2$ that lies b/n 2 and 3.

Ans:-

(4) Using the Regula Falsi method, find the root, correct to three significant figures of the equation $x e^{x} = 2$ that lies between 0 and 1.

Ans: 0.853.

(5) Using the Method of False position find a real root (correct to three decimal places) of the equation $\cos x = 3x - 1$ that lies b/n 0.5 and 1. Here x is in radians. Ans:- 0.607.

(6) Using the Regula Falsi Method, find the root, correct to four decimal places of the equation $x e^{x} = \cos x$ that lies between 0.4 and 0.6 (Here x is in radians) Ans:- 0.5177.

(7) Verify that the equation $\tan x + \tanh x = 0$ where x is in radians, has a root between 2 and 3. Find the 3rd approximation of this root by Regula Falsi Method. Ans: 2.3993.

(8) Using the Regula Falsi Method, find the negative root of $x^3 - 4x + 9 = 0$.

(9) By using the method of false position, find the fourth roots of 12 and correct to three decimal places. Ans:- 1.861

→ Find the real root of the equation $x^3 - 2x - 5 = 0$ using Regula Falsi method.

Sol:- Given that the equation is $f(x) = x^3 - 2x - 5 = 0$.

Let $f(x) = x^3 - 2x - 5$.

$x = 1 \quad f(1) = 1 - 2 - 5 = -6 < 0$

$x = 2 \quad f(2) = 8 - 4 - 5 = -1 < 0$

$x = 3 \quad f(3) = 27 - 6 - 5 = 16 > 0$.

We observe that $f(2)$ and $f(3)$ are of opposite signs.

∴ The given equation does have a real root between 2 and 3.

By Regula Falsi Method.

$$x_{i+1} = \frac{x_{i-1} f(x_i) - x_i f(x_{i-1})}{f(x_i) - f(x_{i-1})}.$$

<u>First Approximation</u> :—

$$i = 1, \quad x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

Let $x_0 = 2 \quad f(x_0) = f(2) = -1$.

$x_1 = 3 \quad f(x_1) = f(3) = 16$.

$$x_2 = \frac{2(16) - 3(-1)}{16 - (-1)} = \frac{35}{17} = 2.05882.$$

$$f(x_2) = f(2.05882) = (2.05882)^3 - 2(2.05882) - 5.$$

$$= -0.39084 < 0$$

We observe that $f(2.05882)$ and $f(3)$ are of opposite signs.

∴ The root lies between 2.05882 and 3.

Second Approximation :—

$$i=2, \quad x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)}$$

$x_1 = 2.05882 \qquad f(x_1) = -0.39084$

$x_2 = 3 \qquad f(x_2) = 16.$

$$x_3 = \frac{(2.05882).16 - 3(-0.39084)}{16 + 0.39084}$$

$$= \frac{34.11364}{16.39084} = 2.08126.$$

$$f(x_3) = f(2.08126) = (2.08126)^3 - 2(2.08126) - 5$$

$$f(x_3) = -0.14724 < 0.$$

We observe that $f(2.08126)$ and $f(3)$ are of opposite signs.

$\therefore$ The root lies between 2.08126 and 3.

Third Approximation :—

$$i=3 \qquad x_4 = \frac{x_2 f(x_3) - x_3 f(x_2)}{f(x_3) - f(x_2)}$$

$x_2 = 2.08126 \quad f(x_2) = f(2.08126) = -0.14724$

$x_3 = 3 \qquad f(x_3) = f(3) = 16$

$$x_4 = \frac{(2.08126)16 - 3(-0.14724)}{16 + 0.14724}$$

$$= \frac{33.74188}{16.14724} = 2.08964$$

$$f(x_4) = f(2.08964) = (2.08964)^3 - 2(2.08964) - 5$$

$$= -0.05467 < 0.$$

We observe that $f(2.08964)$ and $f(3)$ are of opposite signs.

$\therefore$ The root lies between 2.08964 and 3.

Fourth Approximation :—

$i=4 \qquad x_5 = \dfrac{x_3 f(x_4) - x_4 f(x_3)}{f(x_4) - f(x_3)}$

$x_3 = 2.08964 \qquad f(x_3) = -0.05467$

$x_4 = 3 \qquad f(x_4) = 16.$

$$x_5 = \frac{(2.08964).16 - 3(-0.05467)}{16 + 0.05467}$$

$$= \frac{33.59825}{16.05467} = 2.09274$$

$$f(x_5) = f(2.09274) = (2.09274)^3 - 2(2.09274) - 5$$

$$= -0.020198 < 0 .$$

We observe that $f(2.09274)$ and $f(3)$ are of opposite signs.

$\therefore$ The root lies between 2.09274 and 3.

Fifth Approximation :—

$i=5, \qquad x_6 = \dfrac{x_4 f(x_5) - x_5 f(x_4)}{f(x_5) - f(x_4)}$

$x_4 = 2.09274 \qquad f(x_4) = -0.020198$

$x_5 = 3 \qquad f(x_5) = 16 > 0.$

$$x_6 = \frac{(2.09274)16 - 3(-0.020198)}{16 + 0.020198}$$

$$= \frac{33.544434}{16.020198} = 2.09388$$

$$f(x_6) = f(2.09388) = (2.09388)^3 - 2(2.09388) - 5 .$$

$$= -0.007491 < 0 .$$

We observe that $f(2.09388)$ and $f(3)$ are of opposite signs.

$\therefore$ The root lies between 2.09388 and 3.

Sixth Approximation :-

$i=6 \qquad x_7 = \dfrac{x_5 f(x_6) - x_6 f(x_5)}{f(x_6) - f(x_5)}$

$x_5 = 2.09388 \qquad f(x_5) = -0.007491$

$x_6 = 3 \qquad f(x_6) = 16$

$$x_7 = \frac{(2.09388)16 - 3(-0.007491)}{16 + 0.007491}$$

$$= \frac{33.524553}{16.007491} = 2.0943$$

$f(x_7) = f(2.0943) = (2.0943)^3 - 2(2.0943) - 5 = -0.002806$

We observe that $f(2.0943)$ and $f(3)$ are of opposite signs.

$\therefore$ The root lies between 2.0943 and 3.

Seventh Approximation :—

$i=7, \qquad x_8 = \dfrac{x_6 f(x_7) - f(x_6) x_7}{f(x_7) - f(x_6)}$

$x_6 = 2.0943 \qquad f(x_6) = -0.002806$

$x_7 = 3 \qquad f(x_7) = 16$

$$x_8 = \frac{(2.0943)16 - 3(-0.002806)}{16 + 0.002806}$$

$$x_8 = \frac{33.517218}{16.002806} = 2.0945$$

6th and 7th approximations are approximately equal.

$\therefore$ An approximate root of the given equation is 2.0945.

→ obtain Newton Raphson formula to find the square root of N and hence deduce the value of $\sqrt{8}$.

Sol:- Let $x = \sqrt{N}$

$x^2 = N$

$x^2 - N = 0$

Let $f(x) = x^2 - N$

$f'(x) = 2x$

The Newton Raphson formula is $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$$x_{n+1} = x_n - \frac{x_n^2 - N}{2x_n}$$

$$= \frac{2x_n^2 - x_n^2 + N}{2x_n}$$

$$x_{n+1} = \frac{x_n^2 + N}{2x_n}.$$

Which is the Newton Raphson formula for find a square root of N.

To find square root of N = 8 :-

Let $f(x) = x^2 - 8$

$x = 2 \quad f(2) = 4 - 8 = -4 < 0$

$x = 3 \quad f(3) = 9 - 8 = 1 > 0.$

∴ The root lies between 2 and 3.5

Let the initial approximation $x_0 = 2.5$

1st Approximation :-

$n = 0, \quad x_1 = \frac{x_0^2 + 8}{2x_0}$

$$x_1 = \frac{(2.5)^2 + 8}{2(2.5)} = 2.85.$$

2^{nd} Approximation :-

$$n=1, \quad x_2 = \frac{x_1^2+8}{2x_1}$$

$$x_2 = \frac{(2.85)^2+8}{2(2.85)}$$

$$x_2 = 2.8285$$

3^{rd} Approximation :-

$$n=2, \quad x_3 = \frac{x_2^2+8}{2x_2}$$

$$x_3 = \frac{(2.8285)^2+8}{2(2.8285)}$$

$$x_3 = 2.8284 .$$

2^{nd} and 3^{rd} approximations are approximately equal.

$\therefore$ Square root of 8 is equals to 2.8284 approximately

→ Obtain Newton Raphson formula to find the fifth root of N. and hence deduce the value of $\sqrt[5]{5}$.

Sol:- Let $x = \sqrt[5]{N}$.

$$x = N^{1/5}$$

$$x^5 = N \Rightarrow x^5 - N = 0.$$

Let $f(x) = x^5 - N$.

$$f'(x) = 5x^4 .$$

The Newton Raphson formula is $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$.

$$x_{n+1} = x_n - \frac{x_n^5 - N}{5x_n^4}$$

$$x_{n+1} = \frac{4x_n^5 + N}{5x_n^4}$$

Which is the Newton Raphson formula for finding the fifth root of N.

To find fifth root of N = 5 :-

Let $f(x) = x^5 - N$.

$f(x) = x^5 - 5$

$x = 1 \quad f(1) = 1 - 5 = -4 < 0$

$x = 1.2 \quad f(1.2) = (1.2)^5 - 5 = -2.51168 < 0$.

$x = 1.5 \quad f(1.5) = (1.5)^5 - 5 = 2.59375 > 0$.

$\therefore$ The root lies between 1.2 and 1.5

Let the initial approximation $x_0 = 1.3$.

1st Approximation :-

$$n = 0, \quad x_1 = \frac{4x_0^5 + 5}{5x_0^4}$$

$$x_1 = \frac{4(1.3)^5 + 5}{5(1.3)^4} = 1.39013$$

2nd Approximation :—

$$n = 1, \quad x_2 = \frac{4x_1^5 + 5}{5x_1^4}$$

$$x_2 = \frac{4(1.39013)^5 + 5}{5(1.39013)^4} = 1.37988.$$

3rd Approximation :—

$$n = 2, \quad x_3 = \frac{4x_2^5 + 5}{5x_2^4}$$

$$x_3 = \frac{4(1.37988)^5 + 5}{5(1.37988)^4} = 1.37973$$

4th Approximation :-

$n=3, \quad x_4 = \dfrac{4x_3^5+5}{5x_3^4}$

$= \dfrac{4(1.37393)^5+5}{5(1.37393)^4}$

$= 1.37978$

3rd and 4th approximations are approximately equal.

$\therefore$ Fifth root of 5 is equal to 1.37978 approximately.

→ Obtain Newton Raphson formula to find the reciprocal of a number N and hence deduce the value of $\frac{1}{7}$.

Sol: Let $x = \dfrac{1}{N}$.

$N = \dfrac{1}{x} \Rightarrow \dfrac{1}{x} - N = 0.$

Let $f(x) = \dfrac{1}{x} - N.$

$f'(x) = \dfrac{-1}{x^2}.$

The Newton Raphson formula is $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

$$x_{n+1} = x_n - \frac{\frac{1}{x_n} - N}{-\frac{1}{x_n^2}}$$

$$= x_n + \frac{\frac{1-Nx_n}{x_n}}{\frac{1}{x_n^2}}$$

$$= x_n + x_n - Nx_n^2.$$

$$x_{n+1} = 2x_n - Nx_n^2.$$

Which is the Newton Raphson formula for finding the value of reciprocal of N.

To find reciprocal of $N = 7$

Let $f(x) = \frac{1}{x} - 7$

$x = 0.1 \qquad f(0.1) = \frac{1}{0.1} - 7 = 3 > 0$

$x = 1 \qquad f(1) = 1 - 7 = -6 < 0$

∴ The root lies between 0.1 and 1

Let the initial approximation $x_0 = 0.1$

1st Approximation :—

$n = 0, \qquad x_1 = 2x_0 - 7x_0^2$

$x_1 = 2(0.1) - 7(0.1)^2 = 0.13$

2nd Approximation :—

$n = 1, \qquad x_2 = 2x_1 - 7x_1^2$

$x_2 = 2(0.13) - 7(0.13)^2 = 0.1417$

3rd Approximation :—

$n = 2, \qquad x_3 = 2x_2 - 7x_2^2$

$x_3 = 2(0.1417) - 7(0.1417)^2 = 0.1428$

4th Approximation :—

$n = 3 \qquad x_4 = 2x_3^2 - 7x_3^2$

$= 2(0.1428) - 7(0.1428)^2$

$= 0.14286.$

3rd and 4th approximations are approximately equal.

∴ Reciprocal of $7 = 0.14286$.

→ Find a real root of the equation $xe^x - \cos x = 0$ using Newton Raphson Method.

Sol:- Given that the equation $xe^x - \cos x = 0$

Let $f(x) = xe^x - \cos x$

The cos function values measured in radians

$x = 0, \quad f(0) = 0 - \cos 0 = -1 < 0$

$x = 1 \quad f(1) = e^1 - \cos 1 = 2.17 > 0$

∴ The root lies between 0 and 1.

The Newton Raphson formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

1st Approximation :—

$n = 0, \quad x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

$f(x) = xe^x - \cos x \qquad f'(x) = (x+1)e^x + \sin x.$

Let the initial approximation $x_0 = 0.5$.

$f(x_0) = f(0.5) = (0.5)e^{0.5} - \cos(0.5) = -0.05322$

$f'(x_0) = f'(0.5) = (0.5+1)e^{0.5} + \sin(0.5) = 2.9525.$

$$x_1 = 0.5 - \frac{(-0.05322)}{2.9525}$$

$$x_1 = 0.5 + \frac{0.05322}{2.9525} = 0.518025$$

2nd Approximation :-

$n = 1 \qquad x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

At $x_1 = 0.518025$

$f(x_1) = f(0.518025) = (0.518025)e^{0.518025} - \cos(0.518025)$

$= 0.0008144.$

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$$f'(x_1) = f'(0.518025) = (0.518025+1)e^{0.518025} + \sin(0.518025)$$

$$= 3.04348734$$

$$x_2 = 0.518025 - \frac{0.0008144}{3.04348734}$$

$$= 0.518025 - 0.00026759$$

$$x_2 = 0.5178$$

3rd Approximation :—

$$n = 2, \quad x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

At $x_2 = 0.5178$ $\quad f(x_2) = f(0.5178) = (0.5178)e^{0.5178} - \cos(0.5178)$

$$= 0.0001297$$

$$f'(x_2) = f'(0.5178) = (0.5178+1)e^{0.5178} + \sin(0.5178)$$

$$= 3.04234$$

$$x_3 = 0.5178 - \frac{0.0001297}{3.04234}$$

$$= 0.5178 - 0.00004263$$

$$x_3 = 0.51776$$

2nd and 3rd Approximations are approximately equal.

$\therefore$ The root of the equation is 0.51776.

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NEWTON RAPHSON METHOD

(1) Using the Newton Raphson method find a real root, correct to four significant figures, of the equation $x^4 + 2x^2 - 16x + 5 = 0$ which lies near $x = 0$.

Ans:- 0.3265

(2) Apply Newton Raphson method to find an approximate root of the equation $x^3 - 3x - 5 = 0$ which lies near $x = 2$, correct five decimal places.

Ans- 2.27902.

(3) By Newton Raphson method find the real root of the equation $xe^x = 2$ correct to four significant figures.

Ans:- 0.8526.

(4) Using the Newton Raphson method, find the real root, correct to three decimal places, of the equation $\cos x - xe^x = 0$ which lies near $x = 0.5$ (Here x is in radians) correct to four significant figures.

Ans:-

(5) Using the Newton Raphson method, find the root that lies near $x = 4.5$ of the equation $\tan x = x$ correct to four significant figures.

Ans:-

(6) Using the Newton Raphson method find the real root of the equation $x\sin x + \cos x = 0$ near $x = \pi$. upto four decimal places (Here x is in radians)

Ans:-

(7) By Newton Raphson method, find the real root of the equation $x^2 + x + \cos x = 0$ near 0.5. correct to 4 decimal places (Here x is in radians)

(8) By Newton Raphson method, find the real root of the equation $x^3 - 6x + 4 = 0$ near 0.75 correct to 4 decimal places.

(9) obtain Newton Raphson formula to find square root of N and hence deduce the square root of 7.

Ans :- $x_{n+1} = \dfrac{x_n^2 + N}{2x_n}$

(10) obtain Newton Raphson formula to find cube root of N and hence deduce the cube root of 17.

Ans :- $x_{n+1} = \dfrac{2x_n^3 + 12}{3x_n^2}$

(11) obtain Newton Raphson formula to find fifth root of N and hence deduce the fifth root of 5.

Ans :-

(12) obtain Newton Raphson formula to find k^{th} root of positive number N and hence deduce the 7^{th} root of 7.

Ans :- $x_{n+1} = \dfrac{1}{K}\left[(K-1)x_n + \dfrac{N}{x_n^{K-1}}\right]$.

(13) obtain Newton Raphson formula to find the reciprocal of N and hence deduce the value of $\frac{1}{31}$.

Ans :- $x_{n+1} = 2x_n - N x_n^2$

(14) obtain Newton Raphson formula to find the reciprocal of k^{th} root of N and hence deduce the value of $\frac{1}{\sqrt{12}}$.

Ans :- $x_{n+1} = \dfrac{x_n}{R}\left[R+1 - N x_n^K\right]$.

(15) Find an approximate positive root of the equation $e^x \sin x = 1$ using the Newton Raphson method (Here x, is in radians)

Ans :- 0.58853.

(16) Explain Newtons Raphson Method with geometrical interpretation.

(17) Prove that the order of convergence of Newton Raphson method is 2.

Iteration Method :

Iteration is the process in which we perform the same procedure again and again.

In iterative method we can solve the problem by calculating the successive approximations to the solution with an initial guess.

Iterative methods are useful for the problems involving a large no. of variables.

Let $f(x)=0$ be the given equation.

It can be expressed as $x=\phi(x)$.

Let x_0 be an approximate value of the desired root α.

We calculate $x_1=\phi(x_0)$.

The successive approximations are given by.

$$x_2=\phi(x_1)$$
$$x_3=\phi(x_2)$$
$$\cdots$$
$$x_n=\phi(x_{n-1}).$$

We proceed in this way until the two successive approximations are approximately equal.

$x_n=\phi(x_{n-1})$ is called the iterative formula

This Method is convergent if $|\phi'(x)|<1 \;\forall\, x\in I$.

Where I is the interval which contains a root of the equation.

1) By the fixed point iteration process, find the root correct to 3-decimal places of the equation $x = \cos x$ near $x = \frac{\pi}{4}$.

Sol:- Given that the equation is $x = \cos x$.

which is of the form $x = \phi(x)$

where $\phi(x) = \cos x$.

$\phi'(x) = \sin x$.

$|\phi'(x)| = |\sin x| < 1 \quad \forall x$

Since the root is required near $x = \frac{\pi}{4}$.

We take the initial approximation of the root as $x_0 = \frac{\pi}{4} = 0.78571$

The iteration formula is $x_n = \phi(x_{n-1})$.

$n=1$, $x_1 = \phi(x_0) = \cos(0.78571) = 0.70689$.

$n=2$, $x_2 = \phi(x_1) = \cos(0.70689) = 0.76039$

$n=3$ $x_3 = \phi(x_2) = \cos(0.76039) = 0.72457$.

$n=4$, $x_4 = \phi(x_3) = \cos(0.72457) = 0.74878$

$n=5$, $x_5 = \phi(x_4) = \cos(0.74878) = 0.73252$

$n=6$, $x_6 = \phi(x_5) = \cos(0.73252) = 0.74349$

$n=7$ $x_7 = \phi(x_6) = \cos(0.74349) = 0.73611$

$n=8$ $x_8 = \phi(x_7) = \cos(0.73611) = 0.74109$

$n=9$ $x_9 = \phi(x_8) = \cos(0.74109) = 0.73773$.

$n=10$ $x_{10} = \phi(x_9) = \cos(0.73773) = 0.739997$.

$n=11$ $x_{11} = \phi(x_{10}) = \cos(0.739997) = 0.73847$.

$n=12$ $x_{12} = \phi(x_{11}) = \cos(0.73847) = 0.73949$.

. . . .

12^{th} and 13^{th} approximations are approximately equal.

$\therefore$ The root of the equation is 0.739.

(2) Find the positive root of the equation $x^4-x-10=0$ by iteration

Sol:- Given that the equation is $x^4-x-10=0$.

It can be written as $x=\phi(x)$ in many ways such as

$$x = x^4-10 \qquad x=\frac{10}{x^3-1} \qquad x=(x+10)^{\frac{1}{4}}$$

only $x=(x+10)^{1/4}$ satisfies the converge criteria $|\phi'(x)|<1$.

Let $f(x) = x^4-x-10$

$x=1$, $f(1) = -10 < 0$

$x=2$ $f(2) = 16-2-10 = 4>0$.

$\therefore$ The root lies between 1 and 2.

The Iteration formula is $x_n = \phi(x_{n-1})$.

$$\phi(x) = (x+10)^{1/4}$$

$$\phi'(x) = \frac{1}{4}(x+10)^{\frac{-3}{4}}.$$

$$|\phi'(x)| < 1 \quad \forall x \in [1,2].$$

choose $x_0 = 1.5$

$n=1$, $x_1 = \phi(x_0) = (x_0+10)^{\frac{1}{4}} = (1.5+10)^{\frac{1}{4}} = 1.8415$.

$n=2$ $x_2 = \phi(x_1) = (x_1+10)^{1/4} = (1.8415+10)^{1/4} = 1.8550$.

$n=3$ $x_3 = \phi(x_2) = (x_2+10)^{1/4} = (1.8550+10)^{1/4} = 1.8556$.

$n=4$ $x_4 = \phi(x_3) = (x_3+10)^{1/4} = (1.8556+10)^{1/4} = 1.8556$.

x_3 and x_4 are approximately equal.

$\therefore$ The approximate root of the equation is 1.8556.

Geometrical Interpretation of iteration Method :—

The iteration method can be represented geometrically as follows.

Let $x_0, x_1, x_2, \ldots x_n$ denote the successive approximations to the root ξ

Then $x_1 = \phi(x_0)$

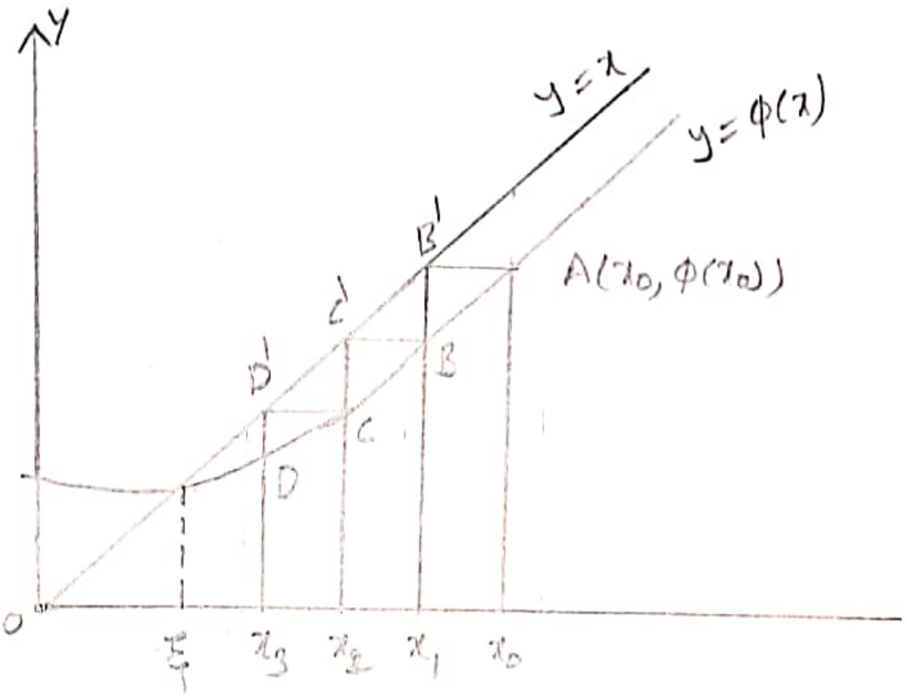
$x_2 = \phi(x_1)$

$x_3 = \phi(x_2)$ etc.

can be viewed as points by the following geometric constructions.

By sketching the line $y=x$ and the curve $y=\phi(x)$ the intersection of these two curves gives the exact root $x=\xi$. Since $|\phi'(x)|<1$, then in the neighbourhood of x_0 the inclination of the curve of $y=\phi(x)$ must be less than $\frac{\pi}{4}$ or 45°

In order to trace the convergence of the iteration method we first draw the ordinate $\phi(x_0)$ parallel to y-axis, which meets the curve $\phi(x)$ at point $A(x_0, \phi(x_0))$ then from point A draw a line parallel to x-axis which meets the line $y=x$ at the points $B'(x_1, \phi(x_0))$, it is noted here that this point B' is the geometric representation of the first Iteration equation $x_1 = \phi(x_0)$.



Now draw a line from B' parallel to y-axis which meets the curve $\phi(x)$ at point $B(x_1, \phi(x_1))$ and again draw a line from B parallel to x-axis which meets the line $y=x$ at c' whose abssicsa is x_2 which gives the second approximation., Continue this process of drawing the lines parallel to the co ordinate axes, we finally approaches to ξ.

ITERATION METHOD

(1) Using the Iteration method, find the real root of the equation $x^3 - x - 1 = 0$ that is near $x = 1$. correct to four decimal places.

Ans:- 1.3247.

(2) By the fixed point iteration method, find the root of the equation $x^3 - 2x - 5 = 0$ that is near $x = 2$. correct to 5 significant figures.

Ans:- 2.0946.

(3) By the iteration process find the root, correct to 5 decimal places, of the equation $x = \cos x$ near $x = \frac{\pi}{4}$.

Ans:- 0.73899.

(4) Using the iteration method, find the root of the equation $3x = 1 + \cos x$ in the interval (0, 1) Here x is in radians.

Ans:- 0.60713.

(5) Evaluate $\sqrt{12}$ and $\frac{1}{\sqrt{12}}$ by using the iteration method.

Ans:- 3.464285, 0.28869.

(6) Solve $x = 1 + \tan^{-1} x$ by using the iteration method.

Ans:- $x = 2.1323$.

(7) Solve $x^3 = 2x + 5$ for a positive by iteration method.

Ans:-

(8) Using the method of iteration find a positive root between 0 and 1 of the equation $x e^x = 1$.

Ans:-

9) Solve $x^2 - 2x + 1 = 0$ using iteration method.

10 Solve $\sin x - 2x + 1 = 0$ busing iteration method by taking $x_0 = 1.5$

11) Solve $x^4 - 12x + 7 = 0$ using iteration method.

12) Solve $x^3 + x^2 - 1 = 0$ for a positive root by iteration method.

Interpolation :-

Interpolation is a method of constructing new data points from a discrete set of known data points.

i.e. Interpolation is the process of finding out the unknown value which lies in the given set of tabulated values.

Extrapolation :-

Extrapolation is the process of finding out the unknown value which lies outside the given set of tabulated values.

Introduction :-

Let $y = f(x)$ $x_0 \le x \le x_n$ be defined. We can find the value of y for all values of x because $y = f(x)$ is defined explicitely.

But without having the explicit definition of $y = f(x)$, it is diffi-cult to find y.

If we have a set of tabular values

x	x_0	x_1	x_2	$\dots$	x_n
y	y_0	y_1	y_2	$\dots$	y_n

Which satisfy $y = f(x)$ then we can find the value of y for the corresponding value of x, by using interpolation.

Finite differences :—

Let $y = f(x)$ be a function in $x_0 \le x \le x_n$, x_i are equally spaced (i.e the difference between x_i and x_{i+1} is same $\forall\, i = 0,1,2 \ldots n-1$)

Then we can recover the values of y for some intermediate values of x in the range $x_0 \le x \le x_n$ by using the differences of $f(x)$

The first finite difference of y is

$$\Delta y = \Delta f(x)$$

$$\Delta y = f(x+\Delta x) - f(x) \text{ where } \Delta x \text{ is the increment in } x.$$

$$\Delta^2 y = \Delta(\Delta y)$$

$$= \Delta\left[f(x+\Delta x) - f(x)\right]$$

$$= \Delta f(x+\Delta x) - \Delta f(x)$$

$$= f(x+2\Delta x) - f(x+\Delta x) - f(x+\Delta x) + f(x)$$

$$\Delta^2 y = f(x+2\Delta x) - 2f(x+\Delta x) + f(x).$$

In general $\Delta^n y = \Delta(\Delta^{n-1} y)$ for $n = 2, 3, 4, \ldots$

Forward Differences :—

Let $y = f(x)$ be the continuous function. Let $y_0, y_1, y_2 \ldots y_n$ be the corrosponding values of y at $x = x_0, x_1, x_2 \ldots x_n$ respectively.

Then the differences $y_1 - y_0$, $y_2 - y_1$, $y_3 - y_2 \ldots y_n - y_{n-1}$

i.e $y_i - y_{i-1}$ $\forall\, i = 1, 2, 3 \ldots n$ are called the first forward differe-nces.

They are denoted by $\Delta y_0, \Delta y_1, \Delta y_2, \ldots \Delta y_{n-1}$

$$\Delta y_0 = y_1 - y_0 \qquad \Delta y_1 = y_2 - y_1 \qquad \Delta y_2 = y_3 - y_2 \qquad \Delta y_{n-1} = y_n - y_{n-1}$$

Where Δ is called the forward difference operator.

The difference of the first forward differences are called second forward differences and they are denoted by $\Delta^2 y_0, \Delta^2 y_1, \Delta^2 y_2 \ldots$

Backward Differences :-

Let $y = f(x)$ be the continuous function. Let $y_0, y_1, y_2 \ldots y_n$ be the corresponding values of y at $x = x_0, x_1, x_2 \ldots x_n$ respectively.

The differences $y_1 - y_0,\ y_2 - y_1,\ y_3 - y_2 \ldots y_n - y_{n-1}$ are called the first backward differences. They are denoted by $\nabla y_1, \nabla y_2, \nabla y_3 \ldots \nabla y_n$ respectively where ∇ is called the backward difference operator.

$$\nabla y_1 = y_1 - y_0 \quad \nabla y_2 = y_2 - y_1 \quad \nabla y_3 = y_3 - y_2 \quad \ldots \quad \nabla y_n = y_n - y_{n-1}.$$

Generally $\nabla y_n = y_n - y_{n-1}$.

The differences of the first backward differences are called second backward differences and they are denoted by $\nabla^2 y_2, \nabla^2 y_3, \ldots \nabla^2 y_n$.

Similarly we can define higher backward differences.

$$\nabla^2 y_2 = \nabla y_2 - \nabla y_1 = y_2 - y_1 - (y_1 - y_0) = y_2 - 2y_1 + y_0.$$

$$\nabla^3 y_3 = \nabla^2 y_3 - \nabla^2 y_2 = y_3 - 3y_2 + 3y_1 - y_0.$$

The Backward Difference Table :-

x	y	∇	∇^2	∇^3
x_0	y_0			
		$\nabla y_1 = y_1 - y_0$		
x_1	y_1		$\nabla^2 y_2 = \nabla y_2 - \nabla y_1$	
		$\nabla y_2 = y_2 - y_1$		$\nabla^3 y_3 = \nabla^2 y_3 - \nabla^2 y_2$
x_2	y_2		$\nabla^2 y_3 = \nabla y_3 - \nabla y_2$	
		$\nabla y_3 = y_3 - y_2$		$\nabla^3 y_4 = \nabla^2 y_4 - \nabla^2 y_3$
x_3	y_3		$\nabla^2 y_4 = \nabla y_4 - \nabla y_3$	
		$\nabla y_4 = y_4 - y_3$		
x_4	y_4			

Similarly we can define higher order forward differences

$$\Delta^2 y_0 = \Delta y_1 - \Delta y_0 = y_2 - y_1 - (y_1 - y_0) = y_2 - 2y_1 + y_0$$

$$\Delta^2 y_1 = \Delta y_2 - \Delta y_1 = y_3 - y_2 - (y_2 - y_1) = y_3 - 2y_2 + y_1$$

$$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0 = (y_3 - 2y_2 + y_1) - (y_2 - 2y_1 + y_0)$$

$$= y_3 - 3y_2 + 3y_1 - y_0$$

The Forward Difference Table :-

x	y	Δ	Δ^2	Δ^3	Δ^4
x_0	y_0				
		$\Delta y_0 = y_1 - y_0$			
x_1	y_1		$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$		
		$\Delta y_1 = y_2 - y_1$		$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0$	
x_2	y_2		$\Delta^2 y_1 = \Delta y_2 - \Delta y_1$		$\Delta^4 y_0 = \Delta^3 y_1 - \Delta^3 y_0$
		$\Delta y_2 = y_3 - y_2$		$\Delta^3 y_1 = \Delta^2 y_2 - \Delta^2 y_1$	
x_3	y_3		$\Delta^2 y_2 = \Delta y_3 - \Delta y_2$		$\Delta^4 y_1 = \Delta^3 y_2 - \Delta^3 y_1$
		$\Delta y_3 = y_4 - y_3$		$\Delta^3 y_2 = \Delta^2 y_3 - \Delta^2 y_2$	
x_4	y_4		$\Delta^2 y_3 = \Delta y_4 - \Delta y_3$		
		$\Delta y_4 = y_5 - y_4$			
x_5	y_5				

Note :- (a) Δ can also be defined as $\Delta f(x) = f(x+h) - f(x)$

(b) $\Delta f(x) = \Delta K = 0$ [$\because$ $f(x) = K$ is a constant function Then difference of a constant function is zero]

(c) $\Delta(u_K + v_K) = \Delta(u_K) + \Delta(v_K)$

(d) $\Delta(u_K v_K) = u_K \Delta(v_K) + v_K \Delta(u_K)$

Central Differences :—

The central difference operator δ defined as

$\delta y_{\frac{1}{2}} = y_1 - y_0 \quad \delta y_{\frac{3}{2}} = y_2 - y_1 \quad \delta y_{\frac{5}{2}} = y_3 - y_2 \quad \ldots \quad \delta y_{\frac{2n-1}{2}} = y_n - y_{n-1}$.

Similarly higher order central differences can be defined as

$$\delta^2 y_1 = \delta y_{\frac{3}{2}} - \delta y_{\frac{1}{2}} = y_2 - 2y_1 + y_0$$

$$\delta^2 y_2 = \delta y_{\frac{5}{2}} - \delta y_{\frac{3}{2}} = y_3 - 2y_2 + y_1$$

x	y	δ	δ^2	δ^3
x_0	y_0			
		$\delta y_{\frac{1}{2}} = y_1 - y_0$		
x_1	y_1		$\delta^2 y_1 = \delta y_{\frac{3}{2}} - \delta y_{\frac{1}{2}}$	
		$\delta y_{\frac{3}{2}} = y_2 - y_1$		$\delta^3 y_{\frac{3}{2}} = \delta^2 y_2 - \delta^2 y_1$
x_2	y_2		$\delta^2 y_2 = \delta y_{\frac{5}{2}} - \delta y_{\frac{3}{2}}$	
		$\delta y_{\frac{5}{2}} = y_3 - y_2$		$\delta^3 y_{\frac{5}{2}} = \delta^2 y_3 - \delta^2 y_2$
x_3	y_3		$\delta^2 y_3 = \delta y_{\frac{7}{2}} - \delta y_{\frac{5}{2}}$	
		$\delta y_{\frac{7}{2}} = y_4 - y_3$		$\delta^3 y_{\frac{7}{2}} = \delta^2 y_4 - \delta^2 y_3$
x_4	y_4		$\delta^2 y_4 = \delta y_{\frac{9}{2}} - \delta y_{\frac{7}{2}}$	
		$\delta y_{\frac{9}{2}} = y_5 - y_4$		
x_5	y_5			

- It is clear from the three tables that in a definite numerical case, the same numbers occur in the same positions whether we use forward, backward or central differences.

Then we obtain $\Delta y_0 = \nabla y_1 = \delta y_{\frac{1}{2}}$

(1) Given that $u_0 = 1$ $u_1 = 11$ $u_2 = 21$ $u_3 = 28$ $u_4 = 29$ find Δu_0

x	u	Δ	Δ^2	Δ^3	Δ^4
0	1				
		10			
			0		
1	11				
				−3	
		10			
					0
			−3		
2	21				
				−3	
		7			
3	28		−6		
		1			
4	29				

(2) Given that $u_0 = 3$ $u_1 = 12$ $u_2 = 81$ $u_3 = 200$ $u_4 = 100$ $u_5 = 8$.

find $\Delta^5 u_0$.

(3) If $f(x) = x^3 + 5x - 7$ form a table of forward difference taking $x = -1, 0, 1, 2, 3, 4, 5$ show that the third differences are constant.

(4) Construct a forward difference table from the following data.

x	10	15	20	25	30	35
y	19.97	21.51	22.47	23.52	24.65	25.89

Write the values of $\Delta f(10)$, $\Delta^2 f(10)$, $\Delta^3 f(15)$, and $\Delta^4 f(15)$

Newton's Forward Interpolation Formula :-

Let $y = f(x)$ be a function. At $x = x_0, x_1, x_2, \ldots x_n$ let the corrospondi values of y be $y_0\ y_1\ y_2 \ldots y_n$. Here x values are equally spaced with common difference h.

Then Newton's forward interpolation formula is given by.

$$y = y_0 + P\Delta y_0 + \frac{P(P-1)}{2!}\Delta^2 y_0 + \frac{P(P-1)(P-2)}{3!}\Delta^3 y_0 + \cdots + \frac{P(P-1)(P-2)\cdots(P-(n-1))}{n!}\Delta^n y_0$$

$$\text{Where } \quad P = \frac{x - x_0}{h}$$

Note :- It is used to interpolate the values of y nearer to the begining of a set of tabular values.

(1) Find the cubic polynomial which takes the values.

$y(0) = 1 \quad y(1) = 0 \quad y(2) = 1 \quad y(3) = 10.$

Sol :- The forward difference table is

x	y	Δ	Δ^2	Δ^3
0	1			
		$-1 = \Delta y_0$		
1	0		$2 = \Delta^2 y_0$	
		1		$6 = \Delta^3 y_0$
2	1		8	
		9		
3	10			

Here $x_0 = 0 \quad y_0 = 1 \quad h = 1$

$$P = \frac{x - x_0}{h} = \frac{x - 0}{1} = x$$

$\Delta y_0 = -1 \quad \Delta^2 y_0 = 2 \quad \Delta^3 y_0 = 6.$

Newton's forward interpolation formula is

$$y = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 .$$

$$y = 1 + x(-1) + \frac{x(x-1)}{2!}2 + \frac{x(x-1)(x-2)}{3!}6$$

$$y = 1 - x + x^2 - x + x^3 - 3x^2 + 2x$$

$$y = x^3 - 2x^2 + 1$$

(2) Find the Newton's forward difference interpolating polynomial for the data.

x	0	1	2	3
y	1	3	7	13

Ans:- $y = x^2 + x + 1$

(3) Construct the difference table for data and then express y as a function of x.

x	0	1	2	3	4
y	3	6	11	18	27

Ans:- $y = x^2 + 2x + 3$

(4) Using Newton's forward interpolation formula, find a polynomial of degree 2 which takes the values.

x	4	6	8	10
y	1	3	8	16

Ans: $\frac{3x^2}{8} - \frac{11x}{4} + 6.$

(5) Find the Newton's forward difference interpolating polynomial for the data

x	1	2	3	4	5	6	7	8
y	1	8	27	64	125	216	343	512

(1) The values of $\sin x$ are given below for different values of x. (7) (a)

Find $\sin 32$.

x	30°	35°	40°	45°	50°
$y=\sin x$	0.5	0.5736	0.6428	0.7071	0.7660

Sol:- The value of $\sin 32$ is nearer to the begining of the table.

$\therefore$ We use Newton's forward difference interpolation formula.

The forward Difference table is

x	y	Δ	Δ^2	Δ^3	Δ^4
$30° = x_0$	$0.5 = y_0$				
		Δy_0 0.0736			
35°	0.5736		$\Delta^2 y_0$ −0.0044		
		0.0692		$\Delta^3 y_0$ −0.0005	
40°	0.6428		−0.0049		$\Delta^4 y_0$ 0
		0.0643		−0.0005	
45°	0.7071		−0.0054		
		0.0589			
50°	0.7660				

The Newton's forward interpolation formula is

$$y = y_0 + P\Delta y_0 + \frac{P(P-1)}{2!}\Delta^2 y_0 + \frac{P(P-1)(P-2)}{3!}\Delta^3 y_0 + \frac{P(P-1)(P-2)(P-3)}{4!}\Delta^4 y_0$$

where $P = \frac{x - x_0}{h}$.

$x_0 = 30,\ y_0 = 0.5 \quad \Delta y_0 = 0.0736 \quad \Delta^2 y_0 = -0.0044 \quad \Delta^3 y_0 = -0.0005 \quad \Delta^4 y_0 = 0$

$h = 5 \quad x = 32 \qquad P = \frac{32-30}{5} = \frac{2}{5} = 0.4$

$$y = 0.5 + (0.4)(0.0736) + \frac{(0.4)((0.4)-1)}{2!}(-0.0044) + \frac{0.4(0.4-1)(0.4-2)}{3!}(-0.0005) + \frac{0.4(0.4-1)(0.4-2)(0.4-3)}{4!}(0)$$

$$y = 0.5 + 0.02944 + 0.000528 - 0.000032 = 0.529936$$

15) Construct the forward difference table for data and then express y as a function of x.

x	0	1	2	3	4
y	1	3	9	31	81

Sol: The Forward difference table is

x	y	Δ	Δ^2	Δ^3	Δ^4
0	1				
		$\Delta y_0 = 2$			
1	3		$4 = \Delta^2 y_0$		
		6		$12 = \Delta^3 y_0$	
2	9		16		0
		22		12	
3	31		28		
		50			
4	81				

$x_0 = 0 \quad y_0 = 1 \quad \Delta y_0 = 2 \quad \Delta^2 y_0 = 4 \quad \Delta^3 y_0 = 12.$

$h = 1 \quad P = \frac{x - x_0}{h} = \frac{x-0}{1} = x$

The Newton's forward interpolation formula is

$$y = y_0 + P\Delta y_0 + \frac{P(P-1)}{2!}\Delta^2 y_0 + \frac{P(P-1)(P-2)}{3!}\Delta^3 y_0$$

$$y = 1 + x \cdot 2 + \frac{x(x-1)}{\cancel{2!}}\cancel{4}^{2} + \frac{x(x-1)(x-2)}{\cancel{3!}}\cancel{12}^{2}$$

$$y = 1 + 2\cancel{x} + 2x^2 - 2\cancel{x} + 2x(x^2 - 3x + 2)$$

$$y = 1 + 2x^2 + 2x^3 - 6x^2 + 4x$$

$$y = 2x^3 - 4x^2 + 4x + 1$$

(2) Given that $\sqrt{12500} = 111.8034$ $\sqrt{12510} = 111.8481$ $\sqrt{12520} = 111.8928$

$\sqrt{12530} = 111.9375$ Find $\sqrt{12516}$

Sol:- The value of $\sqrt{12516}$ is nearer to the begining of the table

$\therefore$ We use the Newton's forward interpolation formula.

The forward difference table is

x	y	Δ	Δ^2	Δ^3
12500	111.8034			
		0.0447		
			0	
12510	111.8481			
		0.0447		0
			0	
12520	111.8928			
		0.0447		
12530	111.9375			

$x_0 = 12500$ $y_0 = 111.8034$ $\Delta y_0 = 0.0447$ $\Delta^2 y_0 = 0$ $\Delta^3 y_0 = 0$.

$h = 10$.

The Newton's forward interpolation formula is

$$y = y_0 + P\Delta y_0 + \frac{P(P-1)}{2!}\Delta^2 y_0 + \frac{P(P-1)(P-2)}{3!}\Delta^3 y_0.$$

y where $P = \frac{x - x_0}{h}$

$x = 12516$, $P = \frac{12516 - 12500}{10} = \frac{16}{10} = 1.6$.

$$y = 111.8034 + 1.6(0.0447) + \frac{1.6(1.6-1)}{2!}(0) + \frac{1.6(1.6-1)(1.6-2)}{3!}(0)$$

$y = 111.8034 + 0.07152$

$y = 111.87492$

$\therefore y = \sqrt{12516} = 111.87492$

(3) From the following table, find the no. of students who obtained less than 45 marks.

Marks	No. of students
30 - 40	31
40 - 50	42
50 - 60	51
60 - 70	35
70 - 80	31

sol:- The forward difference table is

Marks less than x	No. of Students y	Δ	Δ^2	Δ^3	Δ^4
40	31				
		42			
50	73		9		
		51		-25	
60	124		-16		37
		35		12	
70	159		-4		
		31			
80	190				

45 is nearer to the begining of the table.

$\therefore$ we use the Newton's forward interpolation formula

Newton's forward interpolation formula is

$$y = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0.$$

where $p = \frac{x - x_0}{h}$

8

$y_0 = 31 \quad \Delta y_0 = 42 \quad \Delta^2 y_0 = 9 \quad \Delta^3 y_0 = -25 \quad \Delta^4 y_0 = 37.$

$x_0 = 40 \quad x = 45 \quad h = 10. \quad p = \frac{45-40}{10} = \frac{5}{10} = 0.5$

$$y = 31 + (0.5)(42) + \frac{(0.5)(0.5-1)}{2!}(9) + \frac{0.5(0.5-1)(0.5-2)}{3!}(-25)$$
$$\frac{0.5(0.5-1)(0.5-2)(0.5-3)}{4!}(37)$$

$y = 31 + 21 - 1.125 - 1.5625 - 1.4453125$

$y = 47.8672$

$y = 48$ (approximately.

From the data given below, find the number of students whose weight is between 60 and 70.

Weight in Kgs	0–40	40–60	60–80	80–100	100–120
No. of students	250	120	100	70	50

Sol: The cumulative values and the difference table is given by

Weight x	No. of students y	Δ	Δ^2	Δ^3	Δ^4
Below 40 x_0	250 y_0				
		120 Δy_0			
60	370		-20 $\Delta^2 y_0$		
				-10 $\Delta^3 y_0$	
		100			20 $\Delta^4 y_0$
			-30		
80	470				
				10	
		70			
			-20		
100	540				
		50			
120	590				

The Newtons forward interpolation formula is given by

$$y = y_0 + p\Delta y_0 + \frac{p(p-1)}{2}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0$$

where $p = \frac{x - x_0}{h}$.

Here $x_0 = 40$, $x = 70$, $h = 20$, $\Delta y_0 = 120$, $\Delta^2 y_0 = -20$

$\Delta^3 y_0 = -10$, $\Delta^4 y_0 = 20$

$y_0 = 250$.

$$p = \frac{70-40}{20} = 1.5$$

$$y = 250 + (1.5)(120) + \frac{(1.5)(0.5)}{2}(-20) + \frac{(1.5)(0.5)(-0.5)}{3!}(-10) + \frac{(1.5)(0.5)(-0.5)(-1.5)}{4!}(20)$$

$$y = 250 + 180 - 7.5 + 0.625 + 0.46875 = 423.59 = 424$$

No. of students whose weight is between 60 and 70

$$y(70) - y(60) = 424 - 370 = 54$$

NEWTONS FORWARD INTERPOLATION FORMULA.

(1) Using Newton's forward interpolation formula, find a second degree polynomial passes through the points (1, -1) (2, -1) (3, 1) and (4, 5).

Ans:- $x^2 - 3x + 1$

(2) Find a cubic polynomial which takes the following values.

x	0	1.	2	3
y	1	2	1	10

Ans:- $2x^3 - 7x^2 + 6x + 1$

(3) Find a Newton's forward interpolation polynomial for the data.

x	4	6	8	10
y	1	3	8	16

Ans:- $\frac{1}{8}(3x^2 - 22x + 48)$.

(4) Find an interpolating polynomial for the function $f(x)$.

x	0	2	4	6	8	10
y	0	4	56	204	496	980

Ans:- $x^3 - 2x$.

(5) Using the Newton's forward differences formula, find the interpolating polynomial for the function $y = f(x)$ given by $f(0) = 1$, $f(1) = 2$, $f(2) = 1$ $f(3) = 10$. Hence evaluate $f(0.75)$ and $f(-0.5)$.

Ans:- $2x^3 - 7x^2 + 6x + 1$, 2.40625, -4.

(6) Given $f(0) = 1$, $f(1) = 0$ $f(2) = 1$ $f(3) = 10$, find an interpolating polynomial for $f(x)$ using the Newtons forward interpolation formula. Hence evaluate $f(0.4)$. Ans:- $x^3 - 2x^2 + 1$, 0.744

(7) From the data given in the following table, find the number of students who obtained (i) less than 45 marks (ii) between 40 and 45 marks.

Marks	30 – 40	40 – 50	50 – 60	60 – 70	70 – 80
No. of students	31	42	51	35	31

Ans:- 48, 17.

(8) Given $\sin 10^\circ = 0.17365$, $\sin 11^\circ = 0.19081$, $\sin 12^\circ = 0.20791$, $\sin 13^\circ = 0.22493$ (3)

determine $\sin(10^\circ 21')$ using the Newton's forward formula.

Ans:- 0.179377

(9) From the following table of values of $y = f(x)$, find the values of y for $x = 3.25$, $x = 3.5$ and $x = 3.75$.

x	3	4	5	6	7	8	9
y	4.8	8.4	14.5	23.6	36.2	52.8	73.9

Ans:- 5.493, 6.319, 7.285

(10) A function $y = f(x)$ is given by the following table.

x	1.0	1.2	1.4	1.6	1.8	2.0
$y = f(x)$	0.00	0.128	0.544	1.296	2.432	4.00

Find an approximate value of $f(1.1)$.

(11) The following table gives a set of values of $f(x) = \frac{\sin x}{x^2}$.

Using this Table, find an approximate value of $\sin(0.15)$.

x	0.1	0.2	0.3	0.4	0.5
$f(x)$	9.9833	4.9696	3.2836	2.4339	1.9177

Ans:- 0.1495

(12) Given $\log_{10} 654 = 2.8156$, $\log_{10} 656 = 2.8169$, $\log_{10} 658 = 2.8182$, $\log_{10} 660 = 2.8195$

$\log_{10} 662 = 2.821$, find $\log_{10} 655$. Ans:- 2.8162

(13) Estimate the value of $\tan(0.12)$

x	0.10	0.15	0.20	0.25	0.30
$y = \tan x$	0.1003	0.1511	0.2027	0.2553	0.3093

Ans:- 0.1205

(14) Given $f(1) = 3.49$, $f(1.4) = 4.82$, $f(1.8) = 5.96$, $f(2.2) = 6.5$ Using the Newtons forward interpolation formula, find $f(1.6)$

Ans:- 5.4396

Note:- Let the given tabular values of unknown function $y=f(x)$ is

x	x_0	x_1	x_2	x_3	x_i	x_{i+1}	x_{i+2}	x_{n-1}
$y=f(x)$	y_0	y_1	y_2	y_3	y_i	y_{i+1}	y_{i+2}	y_{n-1}

(i) For finding one missing term y_i at $x=x_i$ in given table, we equate $(n-1)^{th}$ forward or backward difference to zero, then we solve the resultant equation. Here $n-1$ = No. of known y values.

(ii) For finding two missing terms y_i and y_{i+j} at $x=x_i$ and $x=x_{i+j}$ in given table we equate $(n-2)^{nd}$ forward or backward differences to zero then we solve the resultant equations.

Here $n-2$ = No. of known y values.

1) Estimate the missing term in the following table

x	0	1	2	3	4
y	1	3	9	–	81

Sol:- The forward difference table is

x	y	Δ	Δ^2	Δ^3	Δ^4
0	1				
		2			
			4		
1	3			$P-19$	
		6			$124-4P$
			$P-15$		
2	9			$105-3P$	
		$P-9$			
			$90-2P$		
3	P				
		$81-P$			
4	81				

Let us consider $\Delta^4 y_0 = 0$

$$124 - 4P = 0$$

$$4P = 124$$

$$P = 31$$

Find the missing values in the following data.

x	45	50	55	60	65
y	3	–	2	–	-2.4

Sol. Let the missing value be a, b. Then the difference table is.

x	y	Δ	Δ^2	Δ^3
45	3			
		$a-3$		
50	a		$5-2a$	
				$3a+b-9$
		$2-a$		
55	2		$a+b-4$	
				$3.6-a-3b$.
		$b-2$		
60	b		$-0.4-2b$	
		$-2.4-b$		
65	-2.4			

As only three entires $y_0\ y_2\ y_4$ are given, y can be represented by a 2nd degree polynomial having third differences as zero.

$$\therefore \quad \Delta^3 y_0 = 0 \quad \Delta^3 y_1 = 0$$

$$3a+b=9, \quad a+3b=3.6$$

Solving these eqn's, we get $a = 2.925 \quad b = 0.225$

II Method :-

As only three entries $y_0 = 3$ $y_2 = 2$ $y_4 = -2.4$ are given, y can be represe-sented by a second degree polynomial having third differences as zero.

$$\Delta^3 y_0 = 0 \quad \text{and} \quad \Delta^3 y_1 = 0$$

$$(E-1)^3 y_0 = 0 \quad \text{and} \quad (E-1)^3 y_1 = 0.$$

$$E^3 y_0 - 3E^2 y_0 + 3E y_0 - y_0 = 0, \quad E^3 y_1 - 3E^2 y_1 + 3E y_1 - y_1 = 0.$$

$$y_3 - 3y_2 + 3y_1 - y_0 = 0, \quad y_4 - 3y_3 + 3y_2 - y_1 = 0.$$

$$y_3 + 3y_1 = 9 \quad 3y_3 + y_1 = 3.6$$

Solving these equations, we get

$$y_1 = 2.925 \quad y_3 = 0.225$$

Find the missing term in the table

x	2	3	4	5	6
y	45	49.2	54.1	—	67.4

Sol: Let the missing term be p. Then the forward difference table is.

x	y	Δ	Δ^2	Δ^2	Δ^4
2	45				
		4.2			
			0.7		
3	49.2			$p-59.7$	
		4.9			$240.2-4p$
			$p-59$		
4	54.1			$180.5-3p$	
		$p-54.1$			
			$121.5-2p$		
5	p				
		$67.4-p$			
6	67.4				

We know that $\Delta^4 y_0 = 0$ i.e. $240.2 - 4p = 0$

$$p = 60.05.$$

II Method :—

As only four entries $y_0\ y_1\ y_2\ y_4$ are given therefore $y = f(x)$ can be repre-
-sented by a third degree polynomial.

$$\therefore\ \Delta^3 y = \text{Constant}.$$

$$\text{or}\quad \Delta^4 y_0 = 0$$

$$\text{i.e.}\ (E-1)^4 y_0 = 0$$

$$E^4 y_0 - 4E^3 y_0 + 6E^2 y_0 - 4E y_0 + y_0 = 0.$$

$$y_4 - 4y_3 + 6y_2 - 4y_1 + y_0 = 0.$$

Let the missing entry y_3 be p so that

$$67.4 - 4p + 6(54.1) - 4(49.2) + 45 = 0$$

$$-4p = -240.2$$

$$p = 60.05$$

INTERPOLATION.

(1) Find the missing value in the following table

(a)

x	0	1	2	3	4
y	1	2	4	–	15

Ans:- 8

(b)

x	1	2	3	4	5
y	2	–	40	83	150

Ans:- 15

(c)

x	1	2	3	4	5
y	15.75	17.9	–	22.75	43.2

Ans:- 1

(2) Find the missing terms in the following table.

(a)

x	1	2	3	4	5	6	7
y	103.4	97.6	122.9	–	179.0	–	195.8

Ans: 154.8575
190.825

(b)

x	6	7	8	9	10	11	12
y	0.77815	–	0.90309	0.95424	1	–	1.07918

Ans: 0.84494
1.04146

(c)

x	45	50	55	60	65
y	3	–	2	–	−2.4

Ans:- 2.325
2.025

(3) If $y_0 = 4$, $y_1 = 8$ $y_2 = 21$ $y_3 = 75$ $y_4 = 32$ $y_5 = 16$ and $y_6 = 10$

find $\Delta^6 y_0$ without forming the Difference table.

Newton's Backward Interpolation formula :—

Let $y = f(x)$ be a function. Let $y_0, y_1, y_2, y_3, \ldots y_n$ be the values of y at $x = x_0, x_1, x_2, \ldots x_n$. These x values are equally spaced with common difference h.

Then the Newton's Backward Interpolation formula is given by

$$y = y_n + P\nabla y_n + \frac{P(P+1)}{2!}\nabla^2 y_n + \frac{P(P+1)(P+2)}{3!}\nabla^3 y_n + \cdots + \frac{P(P+1)\ldots(P+(n-1))}{n!}\nabla^n y_n$$

where $P = \frac{x - x_n}{h}$.

Note:- The Newton's Backward interpolation formula is used for interpolating a values of y nearer to the end of the table of values

(1) Use Newton's Backward interpolation formula to find the polynomial satisfied by (3, 6) (4, 24) (5, 60) and (6, 120)

Sol:- The Backward difference table is :

x	y	∇	∇^2	∇^3
3	$6 = y_0$			
		$18 = \nabla y_1$		
4	$24 = y_1$		$18 = \nabla^2 y_2$	
		$36 = \nabla y_2$		$6 = \nabla^3 y_3$
5	$60 = y_2$		$24 = \nabla^2 y_3$	
		$60 = \nabla y_3$		
6	$120 = y_3$			

The Newton's Backward Difference interpolation formula is

$$y = y_n + P\nabla y_n + \frac{P(P+1)}{2!}\nabla^2 y_n + \frac{P(P+1)(P+2)}{3!}\nabla^3 y_n$$

where $P = \frac{x - x_n}{h}$

$x_n = 6 \quad y_n = 120 \quad \nabla y_n = 60 \quad \nabla^2 y_n = 24 \quad \nabla^3 y_n = 6$

$h = 1 \quad p = \frac{x - x_n}{h} = \frac{x-6}{1} = x-6$

$$y = 120 + (x-6)60 + \frac{(x-6)(x-5)}{2} 24 + \frac{(x-6)(x-5)(x-4)}{6} \cdot 6$$

$$y = x^3 - 3x^2 + 2x.$$

(2) Find y at 9 from the following table.

x	2	5	8	11
y	94.8	87.9	81.3	75.1

Sol:- The value of y at $x=9$ is nearer to the ending of the table.

$\therefore$ We use Newton's Backward interpolation formula.

The Backward Difference Table is.

x	y	∇	∇^2	∇^3
2	$94.8 = y_0$			
		$-6.9 = \nabla y_1$		
5	$87.9 = y_1$		$0.3 = \nabla^2 y_2$	
		$-6.6 = \nabla y_2$		$0.1 = \nabla^3 y_3$
8	$81.3 = y_2$		$0.4 = \nabla^2 y_3$	
		$-6.2 = \nabla y_3$		
11	$75.1 = y_3$			

The Newton's Backward interpolation formula is

$$y = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n$$

where $p = \frac{x - x_n}{h}$

$x_n = 11, \quad y_n = 75.1 \quad \nabla y_n = -6.2 \quad \nabla^2 y_n = 0.4 \quad \nabla^3 y_n = 0.1$

$$p = \frac{x - x_n}{h} = \frac{9-11}{3} = -\frac{2}{3}.$$

$$y = 75.1 - \frac{2}{3}(-6.2) + \frac{-\frac{2}{3}\left(-\frac{2}{3}+1\right)}{2!}(0.4) + \frac{-\frac{2}{3}\left(-\frac{2}{3}+1\right)\left(-\frac{2}{3}+2\right)}{3!}(0.1)$$

$$y = 75.1 + 4.133 - 0.044 - 0.005$$

$$y = 79.184.$$

13) Calculate the value $f(7.5)$ for the table.

x	1	2	3	4	5	6	7	8
$f(x)$	1	8	27	64	125	216	343	512

Sol:- The value of $f(x)$ at $x = 7.5$ is nearer to the ending of the table.

$\therefore$ We use Newton's Backward interpolation formula.

The Backward Difference Table is.

x	y	∇	∇^2	∇^3	∇^4	∇^5
1	1					
		7				
2	8		12			
		19		6		
3	27		18		0	
		37		6		0
4	64		24		0	
		61		6		0
5	125		30		0	
		91		6		0.
6	216		36		0	
		127		6		
7	343		42			
		169				
8	512					

The Newton's Backward interpolation formula is

$$y = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 y_n$$

where $p = \frac{x - x_n}{h}$

$x_n = 8 \quad y_n = 512 \quad h = 1, \quad \nabla y_n = 169 \quad \nabla^2 y_n = 42 \quad \nabla^3 y_n = 6$

$$p = \frac{7.5-8}{1} = \frac{-0.5}{1} = -0.5$$

$$y = 512 + (-0.5)169 + \frac{(-0.5)(-0.5+1)}{2!}42 + \frac{(-0.5)(-0.5+1)(-0.5+2)}{3!}(6)$$

$$y = 512 - 84.5 - 5.25 - 0.375$$

$$y = 421.875$$

(4) In the table, the values of y are consecutive terms of a series of which the number 21.6 is the 6^{th} term. Find the first and tenth terms of the results.

x	3	4	5	6	7	8	9
y	2.7	6.4	12.5	21.6	34.3	51.2	72.9

Sol:- The Difference Table is

x	y	Δ	Δ^2	Δ^3	Δ^4
3	2.7				
		3.7			
4	6.4		2.4		
		6.1		0.6	
5	12.5		3.0		0
		9.1		0.6	
6	21.6		3.6		0
		12.7		0.6	
7	34.3		4.2		0
		16.9		0.6	
8	51.2		4.8		
		21.7			
9	72.9				

To find 10^{th} term :—

$x_n = 9 \quad y_n = 72.9 \quad \nabla y_n = 21.7, \quad \nabla^2 y_n = 4.8 \quad \nabla^3 y_n = 0.6$

$h = 1$

The Newton's Backward interpolation formula is

$$y = y_n + P\nabla y_n + \frac{P(P+1)}{2}\nabla^2 y_n + \frac{P(P+1)(P+2)}{3!}\nabla^3 y_n$$

where $P = \frac{x - x_n}{h}$

$P = \frac{10-9}{1} = 1.$

$$y = 72.9 + (+1)(21.7) + \frac{(1)(1+1)}{2}(4.8) + \frac{1(1+1)(1+2)}{3!}(0.6)$$

$$y = 72.9 + 21.7 + 4.8 + 0.6$$

$$y = 100.$$

To find 1^{st} term :—

$x_0 = 3 \quad y_0 = 2.7, \quad \Delta y_0 = 3.7 \quad \Delta^2 y_0 = 2.4 \quad \Delta^3 y_0 = 0.6.$

$h = 1$

The Newton's forward interpolation formula is

$$y = y_0 + P\Delta y_0 + \frac{P(P-1)}{2!}\Delta^2 y_0 + \frac{P(P-1)(P-2)}{3!}\Delta^3 y_0$$

where $P = \frac{x - x_0}{h}$

$P = \frac{1-3}{1} = -2$

$$y = 2.7 + (-2)\,3.7 + \frac{(-2)(-2-1)}{2!}(2.4) + \frac{(-2)(-2-1)(-2-2)}{3!}(0.6)$$

$$y = 2.7 - 7.4 + 7.2 - 2.4$$

$$y = 0.1$$

14) The area A of a circle of diameter d is given below.

d	80	85	90	95	100.
A	5026	5674	6362	7088	7854

Find approximately the area of circles of diameters 82 and 91.

Ans:- 5280.1056, 6504.1536.

15) Find following data is taken from steam table. Find the pressure at temperature $t = 142^\circ$, $t = 175^\circ$

Temp °C	140	150	160	170	180
Pressure kg/cm^2	3.685	4.854	6.302	8.076	10.225

Ans:- 3.898, 4.100.

NEWTONS BACKWARD INTERPOLATION FORMULA.

(1) Using Newton's backward interpolation formula, find the interpolating polynomial for the function given by the following table.

x	10	11	12	13
$y=f(x)$	21	23	27	33

Ans:- $x^2-19x+111$
29.75

Hence find $f(12.5)$.

(2) Using Newton's backward interpolation formula, find the interpolating polynomial for the function given $y=f(x)$.

$f(0)=1$ $f(1)=2$ $f(2)=1$ $f(3)=10$. Hence find $f(2.5)$

Ans:- $2x^3-7x^2+6x+1$, 3.5

(3) Using Newton's backward interpolation formula, find the interpolating polynomials for the functions given by the following tables.

(a)

x	0	1	2	3
$f(x)$	1	3	7	13

Ans:- x^2+x+1

(b)

x	0	1	2	3	4
$f(x)$	-5	-10	-9	4	35

Ans: x^3-6x-5

(c)

x	-4	-2	0	2	4
$f(x)$	-25	1	3	29	127

Ans:- x^3+3x^2+3x+3.

(4) From the following table, estimate the number of students who obtained marks between 76 and 80.

Marks	36-45	46-55	56-65	66-75	76-85
No. of Students	18	40	64	50	28

Ans:- 16.

(5) Given $f(40)=184$, $f(50)=204$ $f(60)=226$ $f(70)=250$ $f(80)=276$ $f(90)=304$ find $f(85)$. Ans:- 289.75.

(6) The population of a certain town is given by the following table

Year	1961	1971	1981	1991	2001
Population (in thousands)	19.96	39.65	58.81	77.18	94.58

Using Newton's forward and backward interpolation formulas, find the increase in the population from the year 1965 to 1995

Ans:- 27.8796, 84.264, 56.38.

(7) The deflection d measured at various distances x from one end of a cantilever is given by the following table.

x	0.0	0.2	0.4	0.6	0.8	1.0
d	0.00	0.0347	0.1173	0.2160	0.2987	0.3333

Find the value of d for $x = 0.1$ and $x = 0.95$

Ans:- , 0.3306.

(8). Find y when $x = 19.5$, 23.4 and 24.5

x	19	20	21	22	23	24	25
y	91.00	100.25	110.00	120.25	131.00	142.55	154.00

Ans:- , 135.44 ,

(9) Estimate the values of $f(42)$, $f(44)$ and $f(21)$.

x	20	25	30	35	40	45
$f(x)$	354	332	291	260	231	204

Ans:- 219, ,

(10) Find $f(42)$ and $f(84)$ using the following table.

x	40	50	60	70	80	90
$f(x)$	184	204	226	250	276	304

Ans:- 187.84, 287.

Central Difference Interpolation Formula :—

Newton's forward interpolation formula is useful to find the value of $y = f(x)$ at a point which is near the begining of x and the. Newton's backward interpolation formula is useful to find the value of y at a point which is near the ending of x.

The central difference interpolation formula which are most suited for interpolation nearer to the middle of the tabulated set

Gauss Forward Interpolation Formula :—

$$y = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{p(p-1)(p+1)}{3!}\Delta^3 y_{-1} + \frac{p(p-1)(p+2)(p+1)}{4!}\Delta^4 y_{-2} + \ldots$$

Where $p = \dfrac{x - x_0}{h}$

Gauss Forward formula is used to interpolate the values of the function for the value of p such that $0 < p < 1$.

Table :-

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
x_{-3}	y_{-3}						
		Δy_{-3}					
x_2	y_{-2}		$\Delta^2 y_{-3}$				
		Δy_{-2}		$\Delta^3 y_{-3}$			
x_{-1}	y_{-1}		$\Delta^2 y_{-2}$		$\Delta^4 y_{-3}$		
		Δy_{-1}		$\Delta^3 y_{-2}$		$\Delta^5 y_{-3}$	
x_0	y_0		$\Delta^2 y_{-1}$		$\Delta^4 y_{-2}$		$\Delta^6 y_{-3}$
		Δy_0		$\Delta^3 y_{-1}$		$\Delta^5 y_{-2}$	
x_1	y_1		$\Delta^2 y_0$		$\Delta^4 y_{-1}$		
		Δy_1		$\Delta^3 y_0$			
x_2	y_2		$\Delta^2 y_1$				
		Δy_2					
x_3	y_3						

For central difference formula, the central ordinate is taken as y_0 corresponding $x = x_0$.

(1) Find the polynomial which fits the data in the following table using Gauss forward formula.

Sol:

x	3	5	7	9	11
y	6	24	58	108	174

Sol. The Difference table is

x	y	Δ	Δ^2	Δ^3
$3 = x_{-2}$	$6 = y_{-2}$			
		$18 = \Delta y_{-2}$		
$5 = x_{-1}$	$24 = y_{-1}$		$16 = \Delta^2 y_{-2}$	
		$34 = \Delta y_{-1}$		$0 = \Delta^3 y_{-2}$
$7 = x_0$	$58 = y_0$		$16 = \Delta^2 y_{-1}$	
		$50 = \Delta y_0$		$0 = \Delta^3 y_{-1}$
$9 = x_1$	$108 = y_1$		$16 = \Delta^2 y_0$	
		$66 = \Delta y_1$		
$11 = x_2$	$174 = y_2$			

Gauss forward interpolation formula is.

$$y = y_0 + p\Delta y_0 + \frac{(p-1)p}{2!}\Delta^2 y_{-1} + \frac{(p-1)p(p+1)}{3!}\Delta^3 y_{-1}$$

$x_0 = 7$, $y_0 = 58$, $\Delta y_0 = 50$ $\Delta^2 y_{-1} = 16$ $\Delta^3 y_{-1} = 0$.

$$p = \frac{x - x_0}{h} = \frac{x-7}{2}$$

$$y = 58 + \left(\frac{x-7}{\cancel{2}}\right)\cdot\cancel{50}^{25} + \frac{1}{2}\left(\frac{x-7}{2}\right)\left(\frac{x-9}{2}\right)\cdot\cancel{16}^{2}$$

$$y = 58 + 25x - 175 + 2x^2 - 32x + 126$$

$$y = 2x^2 - 7x + 9.$$

(2) Find y_{30} using Gauss forward interpolation formula given that (2)

$y_{21} = 18.4708$ $y_{25} = 17.8144$ $y_{29} = 17.1070$ $y_{33} = 16.3432$ $y_{37} = 15.5154$ (14)

sol: The Difference Table is

x	y	Δ	Δ^2	Δ^3	Δ^4
$21 = x_{-2}$	$18.4708 = y_{-2}$				
		$-0.6564 = \Delta y_{-2}$			
$25 = x_{-1}$	$17.8144 = y_{-1}$		$-0.0510 = \Delta^2 y_{-2}$		
		$-0.7074 = \Delta y_{-1}$		$-0.0054 = \Delta^3 y_{-2}$	
$29 = x_0$	$17.1070 = y_0$		$-0.0564 = \Delta^2 y_{-1}$		$-0.0022 = \Delta^4 y_{-2}$
		$-0.7638 = \Delta y_0$		$-0.0076 = \Delta^3 y_{-1}$	
$33 = x_1$	$16.3432 = y_1$		$-0.064 = \Delta^2 y_0$		
		$-0.8278 = \Delta y_1$			
$37 = x_2$	$15.5154 = y_2$				

The Gauss forward interpolation formula is

$$y = y_0 + p\,\Delta y_0 + \frac{(p-1)p}{2!}\Delta^2 y_{-1} + \frac{(p-1)p(p+1)}{3!}\Delta^3 y_{-1} + \frac{(p-2)(p-1)p(p+1)}{4!}\Delta^4 y_{-2}$$

where $p = \dfrac{x - x_0}{h}$.

$x_0 = 29$, $y_0 = 17.1070$ $\Delta y_0 = -0.7638$ $\Delta^2 y_{-1} = -0.0564$,

$\Delta^3 y_{-1} = -0.0076$ $\Delta^4 y_{-2} = -0.0022$

$x = 30$, $h = 4$ $p = \dfrac{30-29}{4} = \dfrac{1}{4} = 0.25$

$$y = 17.1070 + (0.25)(-0.7638) + \frac{(0.25-1)(0.25)}{2!}(-0.0564) + \frac{(0.25-1)(0.25)(0.25+1)}{3!}(-0.0076)$$
$$+ \frac{(0.25-2)(0.25-1)(0.25)(0.25+1)}{4!}(-0.0022)$$

$$y = 17.1070 - 0.19095 + \frac{(-0.75)(0.25)}{2!}(-0.0564) + \frac{(-0.75)(0.25)(1.25)}{3!}(-0.0076)$$
$$+ \frac{(-0.75)(-1.75)(0.25)(1.25)}{4!}(-0.0022)$$

$= 17.1070 - 0.1905 + 0.0053 + 0.000296875 - 0.000037598$

$y = 16.9817$

(3) Given that $f(2) = 10$, $f(1) = 8$ $f(0) = 5$ $f(-1) = 10$ find $f(\frac{1}{2})$ using Gauss forward interpolation formula.

sol. The Difference Table is.

x	y	Δ	Δ^2	Δ^3
$-1 = x_{-1}$	$10 = y_{-1}$			
		$-5 = \Delta y_{-1}$		
$0 = x_0$	$5 = y_0$		$8 = \Delta^2 y_{-1}$	
		$3 = \Delta y_0$		$-9 = \Delta^3 y_{-1}$
$1 = x_1$	$8 = y_1$		$-1 = \Delta^2 y_0$	
		$2 = \Delta y_1$		
$2 = x_2$	$10 = y_2$			

$x_0 = 0$ $y_0 = 5$ $\Delta y_0 = 3$ $\Delta^2 y_{-1} = 8$ $\Delta^3 y_{-1} = -9$. $h = 1$

The Gauss forward interpolation formula is

$$y = y_0 + P\Delta y_0 + \frac{(P-1)P}{2!}\Delta^2 y_{-1} + \frac{(P-1)P(P+1)}{3!}\Delta^3 y_{-1}.$$

where $P = \frac{x - x_0}{h} = \frac{\frac{1}{2} - 0}{1} = \frac{1}{2}$

$$y = 5 + \frac{1}{2}(3) + \frac{-1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}(8) + \frac{-1}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{1}{3!}(-9)$$

$$y = 5 + 1.5 - 1 + 0.5625$$

$$y = 6.0625$$

GAUSS FORWARD INTERPOLATION FORMULA

(1) Find the polynomial which fits the data in the following table using Gauss forward interpolation formula.

x	3	5	7	9	11
y	6	24	58	108	174

Ans:- $2x^2-7x+9$.

(2) Use the Gauss forward interpolation formula to find $f(3.3)$ from the following table.

x	1	2	3	4	5
$y=f(x)$	15.30	15.10	15.00	14.50	14.00

Ans:- 14.8912.

(3) Using the Gauss forward interpolation formula, find the value of $\log_{10} 347.5$ from the following table.

x	320	330	340	350	360
$y=\log_{10} x$	2.5052	2.5185	2.5315	2.5441	2.5563

Ans:- 2.54099.

(4) Use Gauss forward interpolation formula to find y_{30} given $y_{21}=18.4708$ $y_{25}=17.8144$ $y_{29}=17.1070$, $y_{33}=16.3432$ $y_{37}=15.5154$.

Ans:- 16.9213.

(5) Find the value of e^{-x} when $x=1.725$, 1.7489 and $x=1.775$ from the following table, using the suitable interpolation formulas.

x	1.72	1.73	1.74	1.75	1.76	1.77	1.78
e^{-x}	0.179066	0.177284	0.175520	0.173774	0.172045	0.170333	0.168638

Ans:- , 0.173965 ,

(6) From the following table for the function $y=e^x$, find e^x for $x=1.75$, 1.91 and 2.15 using an appropriate interpolation formula.

x	1.7	1.8	1.9	2.0	2.1	2.2
$y=e^x$	5.4739	6.0496	6.6859	7.3891	8.1662	9.0250

Ans:- , 6.7531 ,

Gauss Backward Interpolation Formula:—

$$y = y_0 + p\,\Delta y_{-1} + \frac{p(p+1)}{2!}\Delta^2 y_{-1} + \frac{(p-1)p(p+1)}{3!}\Delta^3 y_{-2} + \frac{(p-1)p(p+1)(p+2)}{4!}\Delta^4 y_{-2} + \frac{(p-2)(p-1)p(p+1)(p+2)}{5!}\Delta^5 y_{-3} + \cdots$$

$$\text{Where } p = \frac{x - x_0}{h}$$

The Gauss Backward formula is used to interpolate the values for the value of p such that $-1 < p < 0$.

Table:-

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
x_{-3}	y_{-3}						
		Δy_{-3}					
x_{-2}	y_{-2}		$\Delta^2 y_{-3}$				
		Δy_{-2}		$\Delta^3 y_{-3}$			
x_{-1}	y_{-1}		$\Delta^2 y_{-2}$		$\Delta^4 y_{-3}$		
		Δy_{-1}		$\Delta^3 y_{-2}$		$\Delta^5 y_{-3}$	
x_0	y_0		$\Delta^2 y_{-1}$		$\Delta^4 y_{-2}$		$\Delta^6 y_{-3}$
		Δy_0		$\Delta^3 y_{-1}$		$\Delta^5 y_{-2}$	
x_1	y_1		$\Delta^2 y_0$		$\Delta^4 y_{-1}$		
		Δy_1		$\Delta^3 y_0$			
x_2	y_2		$\Delta^2 y_1$				
		Δy_2					
x_3	y_3						

For central difference formula, the central ordinate is taken as y_0 corresponding $x = x_0$.

(1) Use Gauss Backward interpolation formula to find the value of y at $x = 1936$ using the following table.

x	1901	1911	1921	1931	1941	1951.
y	12	15	20	27	39	52

sol:- The difference table is

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5
1901	$12 = y_{-4}$					
		$3 = \Delta y_{-4}$				
1911	$15 = y_{-3}$		$2\ \Delta^2 y_{-4}$			
		$5\ \Delta y_{-3}$		$0\ \Delta^3 y_{-4}$		
1921	$20 = y_{-2}$		$2\ \Delta^2 y_{-3}$		$3\ \Delta^4 y_{-4}$	
		$7\ \Delta y_{-2}$		$3\ \Delta^3 y_{-3}$		$\Delta^5 y_{-4}$ -10.
1931	$27 = y_{-1}$		$5\ \Delta^2 y_{-2}$		$-7\ \Delta^4 y_{-3}$	
		$12\ \Delta y_{-1}$		$-4\ \Delta^3 y_{-2}$		
1941	$39 = y_0$		$1\ \Delta^2 y_{-1}$			
		$13\ \Delta y_0$				
1951	$52 = y_1$					

The Gauss Backward interpolation formula is

$$y = y_0 + p\Delta y_{-1} + \frac{p(p+1)}{2!}\Delta^2 y_{-1} + \frac{(p-1)p(p+1)}{3!}\Delta^3 y_{-2}.$$

where $p = \frac{x - x_0}{h}$

$x_0 = 1941 \quad y_0 = 39 \quad \Delta y_{-1} = 12 \quad \Delta^2 y_{-1} = 1 \quad \Delta^3 y_{-2} = -4$

$x = 1936 \quad p = \frac{x - x_0}{h} = \frac{1936 - 1941}{10} = \frac{-5}{10} = -0.5$

$h = 10$

$$y = 39 + (-0.5)\,12 + \frac{(-0.5)(-0.5+1)}{2!}(1) + \frac{(-0.5-1)(-0.5)(-0.5+1)}{3!}(-4)$$

$$y = 39 - 6 + 0.125 - 0.25$$

$$y = 32.625$$

(2) Use Gauss Backward interpolation formula find $y(8)$ form the following table.

x	0	5	10	15	20	25
y	7	11	14	18	24	32

Sol:- The difference table is.

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5
x_{-2} 0	7 y_{-2}					
		4 Δy_{-2}				
x_{-1} 5	$11 = y_{-1}$		-1 $\Delta^2 y_{-2}$			
		3 Δy_{-1}		2 $\Delta^3 y_{-2}$		
$x_0 = 10$	$y_0 = 14$		1 $\Delta^2 y_{-1}$		-1 $\Delta^4 y_{-2}$	
		4 Δy_0		1 $\Delta^3 y_{-1}$		0 $\Delta^5 y_{-2}$
x_1 15	18 y_1		2 $\Delta^2 y_0$		-1 $\Delta^4 y_{-1}$	
		6 Δy_1		0 $\Delta^3 y_0$		
20 x_2	24 y_2		2 $\Delta^2 y_1$			
		8 Δy_2				
25 x_3	32 y_3					

The Gauss Backward interpolation formula is

$$y = y_0 + p\,\Delta y_{-1} + \frac{p(p+1)}{2!}\Delta^2 y_{-1} + \frac{(p-1)p(p+1)}{3!}\Delta^3 y_{-2} + \frac{(p-1)p(p+1)(p+2)}{4!}\Delta^4 y_{-2}$$

Where $p = \frac{x - x_0}{h}$

$x_0 = 10 \quad y_0 = 14 \quad \Delta y_{-1} = 3 \quad \Delta^2 y_{-1} = 1 \quad \Delta^3 y_{-2} = 2$

$\Delta^4 y_{-2} = -1 \quad h = 5 \quad x = 8$

$$p = \frac{x - x_0}{h} = \frac{8-10}{5} = \frac{-2}{5} = -0.4$$

$$y = 14 + (-0.4)(3) + \frac{(-0.4)(-0.4+1)}{2!}(1) + \frac{(-0.4)(-0.4+1)(-0.4-1)}{3!} 2$$
$$+ \frac{(-0.4+2)(-0.4)(-0.4+1)(-0.4-1)}{4!}(-1)$$

$$y = 14 - 1.2 + \frac{(-0.4)(0.6)}{2} + \frac{(0.4)(0.6)(-1.4)}{3!} \cdot 2$$
$$+ \frac{(1.6)(-0.4)(0.6)(-1.4)}{4!}(-1)$$

$$y = 14 - 1.2 - 0.12 + 0.112 - 0.0224$$

$$y = \cancel{12.29}\ 12.7696$$

Disadvantages of backward difference interpolation :-

(i) In Gauss Backward interpolation we can not find polynomia for the given data.

(ii) Newton Backward interpolation does not gives exact accurac for central values.

GAUSS BACKWARD INTERPOLATION FORMULA.

(1) From the following table for the function $y = f(x)$, find y at $x = 32$ using the Gauss backward interpolation formula.

x	25	30	35	40
$y = f(x)$	0.2707	0.3027	0.3386	0.3794

Ans:- 0.3165.

(2) Given that $\sqrt{6500} = 80.6226$ $\sqrt{6510} = 80.6846$ $\sqrt{6520} = 80.7466$. $\sqrt{6530} = 80.8084$. Find $\sqrt{6516}$ by using the Gauss backward interpolation formula. Ans:- 80.7218.

(3) Apply Gauss backward interpolation formula to find y when $x = 25$ from the following table.

x	20	24	28	32
y	2854	3162	3544	3992

Ans:- 3251.25.

(4) For the following data estimate $f(1.720)$, $f(2.68)$ and $f(2.36)$ using an appropriate difference formulae.

x	1.6	1.8	2.0	2.2	2.4	2.6	2.8	3.0
y	0.0495	0.0605	0.0739	0.0903	0.1102	0.1346	0.1644	0.2009.

Ans:-

(5) From the following table for the function $y = f(x)$, find y when $x = 1.35$, $x = 1.15$ and $x = 1.9$ using an appropriate difference formulae.

x	1	1.2	1.4	1.6	1.8	2
y	0.0	−0.112	−0.016	0.336	0.992	2.0

Ans:- , −0.062,

(6) The following table gives the population y (in lakhs) of a certain city in the year x. Find by using an appropriate formula the population in the years 1965, 1985 and 2005.

x	1960	1970	1980	1990	2000	2010
y	12	15	20	27	39	52.

Stirling's Formula :—

Gauss forward difference formula is given by

$$y = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1} + \frac{(p+1)p(p-1)(p-2)}{4!}\Delta^4 y_{-2} + \cdots \quad (1)$$

Gauss backward difference formula is given by

$$y = y_0 + p\Delta y_{-1} + \frac{p(p+1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-2} + \frac{(p+2)(p+1)p(p-1)}{4!}\Delta^4 y_{-2} + \cdots \quad (2)$$

Taking the average of (1) and (2), we will get stirling's formula.

$$\frac{y+y}{2} = \frac{y_0+y_0}{2} + \frac{p\Delta y_0 + p\Delta y_{-1}}{2} + \frac{1}{2}\left[\frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{p(p+1)}{2!}\Delta^2 y_{-1}\right]$$
$$+ \frac{1}{2}\left[\frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-2}\right] +$$
$$\frac{1}{2}\left[\frac{(p+1)p(p-1)(p-2)}{4!}\Delta^4 y_{-2} + \frac{(p+2)(p+1)p(p-1)}{4!}\Delta^4 y_{-2}\right] + \cdots$$

$$y = y_0 + \frac{p}{2}(\Delta y_0 + \Delta y_{-1}) + \frac{p}{2}\left[\frac{p-1}{2} + \frac{p+1}{2}\right]\Delta^2 y_{-1} + \frac{p(p-1)(p+1)}{2 \cdot 3!}\left[\Delta^3 y_{-1} + \Delta^3 y_{-2}\right]$$
$$+ \frac{1}{2} \cdot \frac{p(p-1)(p+1)}{4!}\left[p-2+p+2\right]\Delta^4 y_{-2} + \cdots$$

$$y = y_0 + \frac{p}{2}\left[\Delta y_0 + \Delta y_{-1}\right] + \frac{p^2}{2!}\Delta^2 y_{-1} + \frac{p(p^2-1)}{3!}\left[\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right]$$
$$+ \frac{p^2(p^2-1)}{4!}\Delta^4 y_{-2} + \cdots$$

This is stirling's formula, it gives the most accurate result for $-0.25 < p \leq 0.25$.

Therefore we have to choose x_0 such that p satisfies this inequality.

(1) Use stirling's formula to find y_{32} from the following table.

$y_{20} = 14.035$, $y_{25} = 13.674$ $y_{30} = 13.257$, $y_{35} = 12.734$

$y_{40} = 12.089$ $y_{45} = 11.309$.

Sol: The forward difference table is.

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5
x_{-2} 20	14.035 y_{-2}					
		Δy_{-2} −0.361				
x_{-1} 25	13.674 y_{-1}		$\Delta^2 y_{-2}$ −0.056			
		Δy_{-1} −0.417		$\Delta^3 y_{-2}$ −0.05		
$x_0 = 30$	13.257 y_0		$\Delta^2 y_{-1}$ −0.106		$\Delta^4 y_{-2}$ 0.034	
		Δy_0 −0.523		$\Delta^3 y_{-1}$ −0.016		$\Delta^5 y_{-2}$ −0.031
x_1 35	12.734 y_1		$\Delta^2 y_0$ −0.122		$\Delta^4 y_{-1}$ 0.003	
		Δy_1 −0.645		$\Delta^3 y_0$ −0.013		
x_2 40	12.089 y_2		$\Delta^2 y_1$ −0.135			
		Δy_2 −0.78				
x_3 45	11.309 y_3					

The stirling's formula is given by

$$y = y_0 + \frac{p}{2}(\Delta y_0 + \Delta y_{-1}) + \frac{p^2}{2!}\Delta^2 y_{-1} + \frac{p(p^2-1)}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) +$$

$$\frac{p^2(p^2-1)}{4!}\Delta^4 y_{-2} + \dots$$

Here $x = 32$ $x_0 = 30$ $h = 5$ $p = \frac{x - x_0}{h}$

$$p = \frac{x - x_0}{h} = \frac{32-30}{5} = 0.4.$$

$y_0 = 13.257$ $\Delta y_0 = -0.523$ $\Delta y_{-1} = -0.417$.

$\Delta^2 y_{-1} = -0.106$ $\Delta^3 y_{-2} = -0.05$ $\Delta^3 y_{-1} = -0.016$

$\Delta^4 y_{-2} = -0.034$ $\Delta^4 y_{-1} = 0.003$.

$$y = 13.257 + \frac{0.4}{2}(-0.523 - 0.417) + \frac{(0.4)^2}{2}(-0.106)$$

$$+ \frac{(0.4)((0.4)^2 - 1)}{3!}\left[\frac{-0.05 - 0.016}{2}\right] + \frac{(0.4)^2((0.4)^2-1)}{4!}(-0.034)$$

$$y = 13.257 - 0.188 - 0.00848 + 0.001848 - 0.0001904$$

$$y = 13.062$$

(2) Use stirling's formula to find y_{28}, given that $y_{20} = 49225$
$y_{25} = 48316$, $y_{30} = 47236$, $y_{35} = 45926$ $y_{40} = 44306$.

Ans :- 47691.82

STIRLING'S FORMULA.

(1) Using the stirling's formula, find $f(31)$ from the following table

x	20	25	30	35	40
$f(x)$	49.225	48.316	47.236	45.296	44.306

Ans:- 46.895

(2) Apply stirling's formula to find $y_{13.8}$ from the following table.

x	10	12	14	16	18
y_x	0.240	0.281	0.318	0.352	0.384.

Ans:- 0.31445

(3) From the following table for the function $y = \tan x$, find $\tan 14'$ and $\tan 16'$ using the stirling's formula.

x	0	5	10	15	20	25
$y = \tan x$	0	0.0875	0.1763	0.2679	0.3640	0.4663.

Ans:- 0.2493, 0.2867.

(4) Apply stirling's formula to find $\log_{10} 337.5$ from the following table.

x	310	320	330	340	350	360
$\log_{10} x$	2.49136	2.50515	2.51851	2.53148	2.54407	2.55630

Ans:- 2.52828.

(5) Given $\sqrt{1} = 1$, $\sqrt{1.05} = 1.0247$ $\sqrt{1.10} = 1.0488$ $\sqrt{1.15} = 1.0724$ $\sqrt{1.20} = 1.0954$ $\sqrt{1.25} = 1.1180$ $\sqrt{1.30} = 1.1402$ find $\sqrt{1.12}$ using the stirling's formula.

Ans:- 1.0583.

(6) Employ stirling's formula to find $y_{11.8}$ and $y_{12.2}$ from the following table

x	10	11	12	13	14
y_x	0.23967	0.28060	0.31788	0.35209	0.38368

Ans:- , 0.32497.

Interpolation with unevenly spaced points :-

Lagrange's Interpolation Formula :-

Let $x_0\ x_1\ x_2 \dots x_n$ be the $(n+1)$ values of x which are not necessarily equally spaced. Let $y_0, y_1, y_2 \dots y_n$ be the corresponding values of $y=f(x)$. Let the polynomial of degree n for the function $y=f(x)$ passing through the $(n+1)$ points $(x_0, f(x_0))\ (x_1, f(x_1))\ (x_2, f(x_2)) \dots (x_n, f(x_n))$

The Lagrange's interpolation formula is given by

$$y = f(x) = \frac{(x-x_1)(x-x_2)\dots(x-x_n)}{(x_0-x_1)(x_0-x_2)\dots(x_0-x_n)} f(x_0) + \frac{(x-x_0)(x-x_2)\dots(x-x_n)}{(x_1-x_0)(x_1-x_2)\dots(x_1-x_n)} f(x_1) + \dots$$

$$+ \frac{(x-x_0)(x-x_1)\dots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\dots(x_n-x_{n-1})} f(x_n).$$

Find the unique polynomial $P(x)$ of degree 2 or less such that $P(1)=1$ $P(3)=27$ $P(4)=64$ using Lagrange interpolation formula.

Sol:- Given that $x_0=1 \quad x_1=3 \quad x_2=4$

$y_0=1 \quad y_1=27 \quad y_2=64.$

The Lagrange's Interpolation formula is given by.

$$y = f(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2).$$

$$= \frac{(x-3)(x-4)}{(1-3)(1-4)}.1 + \frac{(x-1)(x-4)}{(3-1)(3-4)}.27 + \frac{(x-1)(x-3)}{(4-1)(4-3)}.64$$

$$= \frac{1}{6}\left[48x^2 - 114x + 72\right]$$

$$= 8x^2 - 19x + 12$$

Using Lagrange's interpolation formula, find the form of the fun. $f(x)$ from the following table.

x	0	1	3	4
$f(x)$	-12	0	12	24

sol:- Here $x_0 = 0$ $x_1 = 1$ $x_2 = 3$ $x_3 = 4$

$f(x_0) = -12$ $f(x_1) = 0$ $f(x_2) = 12$, $f(x_3) = 24$.

The values of x are unequally spaced so we apply Lagranges interpolation formula.

Lagranges interpolation formula is given by

$$y = f(x) = \frac{(x-x_1)(x-x_2)\ldots(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} f(x_0) + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} f(x_1)$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} f(x_2) + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} f(x_3).$$

$$y = f(x) = \frac{(x-1)(x-3)(x-4)}{(-1)(-3)(-4)}(-12) + \text{\_\_\_\_\_\_\_\_\_\_} 0$$

$$+ \frac{(x-0)(x-1)(x-4)}{(3)(2)(-1)} 12 + \frac{(x-0)(x-1)(x-3)}{(4)(3)(1)} 24.$$

$$y = f(x) = (x-1)(x-3)(x-4) - 2x(x-1)(x-4) + 2x(x-1)(x-3)$$

$$\therefore y = x^3 - 6x^2 + 17x - 12$$

Using Lagrange's interpolation formula, express $\dfrac{3x^2+x+1}{(x-1)(x-2)(x-3)}$ as sum of partial fractions.

sol: Let $f(x) = 3x^2+x+1$

Take $(x-1)(x-2)(x-3) = 0$

$x = 1, 2, 3$

$x_0 = 1 \quad x_1 = 2 \quad x_2 = 3$

$f(x_0) = 5 \quad f(x_1) = 15 \quad f(x_2) = 31.$

By Lagrange's interpolation formula.

$$y = f(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2)$$

$$3x^2+x+1 = \frac{(x-2)(x-3)}{(-1)(-2)} \cdot 5 + \frac{(x-1)(x-3)}{1 \cdot (-1)} 15 + \frac{(x-1)(x-2)}{2 \cdot 1} (31)$$

Divide with $(x-1)(x-2)(x-3)$, we get

$$\frac{3x^2+x+1}{(x-1)(x-2)(x-3)} = \frac{5}{2(x-1)} - \frac{15}{x-2} + \frac{31}{2(x-3)}.$$

LAGRANGE'S INTERPOLATION FORMULA.

(1) Find the interpolating polynomial for the data given in the following table.

x	0	1	4	5
y	4	3	24	39

Ans:- $2x^3 - 3x + 4$.

(2) Using Lagrange's interpolation formula, fits a polynomial to the following data. Hence find $y(1)$

x	-1	0	2	3
y	-8	3	1	12

Ans:- $2x^3 - 6x^2 + 3x + 3$, Ans:- $y(1) = 2$.

(3) Find the parabola passing through the points $(0,1)$ $(1,3)$ and $(3,55)$. using the Lagrange's interpolation formula.

Ans:- $8x^2 - 6x + 1$.

(4) A curve passes through the points $(0,18)$ $(1,10)$ $(3,-18)$ and $(6,90)$ Find the slope of the curve at $x=2$.

Ans:- $y = 2x^3 - 10x^2 + 18$, $y'(2) = -16$.

(5) obtain the third degree polynomial passing through the four points given below. Hence estimate $f(2.5)$ and $\int_1^2 f(x)\,dx$.

x	1	1.5	2.0	2.8
$f(x)$	3	3.375	5	12.0172

(6) Find $y(5)$, given that $y(0)=1$ $y(1)=3$ $y(3)=13$ $y(8)=123$ using Lagrange's formula.

(7) Using Lagrange's interpolation formula, find y when $x=10$.

x	2	3	8	14
y	94.8	87.9	81.3	68.7

Ans:- 74.925

(8) Given $f(0) = -18$, $f(1) = 0$, $f(3) = 0$ $f(5) = -248$ $f(6) = 0$ $f(9) = 13104$. find $f(2)$ using Lagrange's interpolation formula. Ans:- 28.

Inverse Interpolation :—

So far, given a set of values of x and y, we have been finding the values of y corresponding to a certain value of x. On the other hand, the process of estimating the value of x for a value of y (which is not in the table) is called the inverse interpolation.

Lagrange's formula is merely a relation between two variables either of which may be taken as the independent variable.

Therefore, on interchanging x and y in the Lagrange's formula, we obtain.

$$x = f(y) = \frac{(y-y_1)(y-y_2)\cdots(y-y_n)}{(y_0-y_1)(y_0-y_2)\cdots(y_0-y_n)} x_0 + \frac{(y-y_0)(y-y_2)\cdots(y-y_n)}{(y_1-y_0)(y_1-y_2)\cdots(y_1-y_n)} x_1 + \cdots$$

$$+ \frac{(y-y_0)(y-y_1)\cdots(y-y_{n-1})}{(y_n-y_0)(y_n-y_1)\cdots(y_n-y_{n-1})} x_n .$$

(1) Find the value of x when $y = 0.3$ by applying Lagrange's formula inversely.

x	0.4	0.6	0.8
y	0.3683	0.3332	0.2897

Sol: Here $x_0 = 0.4$ $\quad x_1 = 0.6$ $\quad x_2 = 0.8$

$y_0 = 0.3683$ $\quad y_1 = 0.3332$ $\quad y_2 = 0.2897$.

$y = 0.3$.

Inverse Lagranges interpolation formula is.

$$x = \frac{(y-y_1)(y-y_2)}{(y_0-y_1)(y_0-y_2)} x_0 + \frac{(y-y_0)(y-y_2)}{(y_1-y_0)(y_1-y_2)} x_1 + \frac{(y-y_0)(y-y_1)}{(y_2-y_0)(y_2-y_1)} x_2$$

Sub. the given values in above formula, we get

$$x = \frac{(0.3-0.3332)(0.3-0.2897)}{(0.3683-0.3332)(0.3683-0.2897)}(0.4) +$$

$$\frac{(0.3-0.3683)(0.3-0.2897)}{(0.3332-0.3683)(0.3332-0.2897)}(0.6) +$$

$$\frac{(0.3-0.3683)(0.3-0.3332)}{(0.2897-0.3683)(0.2897-0.3332)}(0.8)$$

$x = -0.0495 + 0.2764 + 0.5305$

$x = 0.7574$.

INVERSE LAGRANGE'S INTERPOLATION.

(1) Given that $f(10) = 16.35$ $f(15) = 14.88$ $f(10) = 13.59$ $f(5) = 12.46$ find x when $f(x) = 14$.

Ans :- 8.33686.

(2) Given that $x_0 = 3$, $x_1 = 5$ $x_2 = 7$ $x_3 = 9$ $x_4 = 11$ $y_0 = 6$ $y_1 = 24$ $y_2 = 58$ $y_3 = 108$ $y_4 = 174$ Find x when $y = 100$.

Ans:- 8.65471

(3) Use Lagranges formula inversly to obtain the value of x when $y = 85$ from the following table.

x	2	5	8	14
y	94.8	87.9	81.3	68.7

Ans:- 6.30383

(4) Find x when $f(x) = 163$ from the following table. Ans:- 82.8

x	80	82	84	86	88
$f(x)$	134	154	176	200	221

(5) Find the value of x when $y = 13.6$ from the following table.

x	30	35	40	45	50
y	15.9	14.9	14.1	13.3	12.5

Ans:- 43.1.

(6) Given that $f(10) = 1754$ $f(15) = 2648$ $f(20) = 3564$ find x when $f(x) = 3000$

Ans:- 16.9.

(7) Find the value of x when $y = 0.3$ by applying Lagrange's formula inversly.

x	0.4	0.6	0.8
y	0.3683	0.3332	0.2897.

Divided Differences :—

In the construction of finite difference tables, x is assumed to be equally spaced. If x is not equally spaced, Lagrange's formula is used to find the unknown value from the table. If an another interpolation point is added to the tabulated data, then the Lagrangian coefficients are to be recalculated which results a different Lagrange's polynomial of higher degree. This difficulty will overcome by taking the Newton's divided differences.

Newton's Divided difference formula :—

Let $y = f(x)$ be a function.

Let $f(x_0), f(x_1), f(x_2) \ldots f(x_n)$ be the values corresponding to $x = x_0, x_1, x_2, \ldots x_n$. Where the values of x are not equally spaced.

The Newton's divided difference formula is

$$f(x) = f(x_0) + (x-x_0) f(x_0,x_1) + (x-x_0)(x-x_1) f(x_0,x_1,x_2) +$$
$$(x-x_0)(x-x_1)(x-x_2) f(x_0,x_1,x_2,x_3) + \ldots + (x-x_0)(x-x_1)\ldots(x-x_{n-1}) f(x_0,x_1,\ldots x_n)$$

Newton's divided difference table is.

x	$y = f(x)$	Δ	Δ^2	Δ^3
x_0	$f(x_0)$			
		$f(x_0, x_1) = \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}$		
x_1	$f(x_1)$		$f(x_0, x_1, x_2) = \dfrac{f(x_1, x_2) - f(x_0, x_1)}{x_2 - x_0}$	$f(x_0, x_1, x_2, x_3) = \dfrac{f(x_1, x_2, x_3) - f(x_0, x_1, x_2)}{x_3 - x_0}$
		$f(x_1, x_2) = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$		
x_2	$f(x_2)$		$f(x_1, x_2, x_3) = \dfrac{f(x_2, x_3) - f(x_1, x_2)}{x_3 - x_1}$	$f(x_1, x_2, x_3, x_4) = \dfrac{f(x_2, x_3, x_4) - f(x_1, x_2, x_3)}{x_4 - x_1}$
		$f(x_2, x_3) = \dfrac{f(x_3) - f(x_2)}{x_3 - x_2}$		
x_3	$f(x_3)$		$f(x_2, x_3, x_4) = \dfrac{f(x_3, x_4) - f(x_2, x_3)}{x_4 - x_2}$	$f(x_2, x_3, x_4, x_5) = \dfrac{f(x_3, x_4, x_5) - f(x_2, x_3, x_4)}{x_5 - x_2}$
		$f(x_3, x_4) = \dfrac{f(x_4) - f(x_3)}{x_4 - x_3}$		
x_4	$f(x_4)$		$f(x_3, x_4, x_5) = \dfrac{f(x_4, x_5) - f(x_3, x_4)}{x_5 - x_3}$	$f(x_3, x_4, x_5, x_6) = \dfrac{f(x_4, x_5, x_6) - f(x_3, x_4, x_5)}{x_6 - x_3}$
		$f(x_4, x_5) = \dfrac{f(x_5) - f(x_4)}{x_5 - x_4}$		
x_5	$f(x_5)$		$f(x_4, x_5, x_6) = \dfrac{f(x_5, x_6) - f(x_4, x_5)}{x_6 - x_4}$	
		$f(x_5, x_6) = \dfrac{f(x_6) - f(x_5)}{x_6 - x_5}$		
x_6	$f(x_6)$			

Compute $f(3)$ using Newton's divided difference formula from the following table

x	1	2	4	8	10
$f(x)$	0	1	5	21	27

Sol:- $x_0 = 1 \quad x_1 = 2 \quad x_2 = 4 \quad x_3 = 8 \quad x_4 = 10$

$f(x_0) = 0 \quad f(x_1) = 1 \quad f(x_2) = 5 \quad f(x_3) = 21 \quad f(x_4) = 27.$

Here the values of x are un equally spaced.

so we apply Newton's divided difference formula.

Newton's divided difference formula is

$$f(x) = f(x_0) + (x-x_0) f(x_0, x_1) + (x-x_0)(x-x_1) f(x_0, x_1, x_2) + (x-x_0)(x-x_1)(x-x_2)\cancel{(x-x_3)} f(x_0, x_1, x_2, x_3) + (x-x_0)(x-x_1)(x-x_2)(x-x_3) f(x_0, x_1, x_2, x_3, x_4).$$

The divided difference table is

x	$f(x)$	Δ	Δ^2		
1	0				
		$\frac{1-0}{2-1} = 1$			
			$\frac{2-1}{4-1} = \frac{1}{3}$		
2	1				
				$\frac{\frac{1}{3}-\frac{1}{3}}{8-1} = 0$	
		$\frac{5-1}{4-2} = 2$			
			$\frac{4-2}{8-2} = \frac{1}{3}$		$\frac{\frac{-1}{16}-0}{10-1} = \frac{-1}{144}$
4	5				
				$\frac{\frac{-1}{6}-\frac{1}{3}}{10-2} = \frac{-1}{16}$	
		$\frac{21-5}{8-4} = 4$			
			$\frac{3-4}{10-4} = \frac{-1}{6}$		
8	21				
		$\frac{27-21}{10-8} = 3$			
10	27				

$f(x_0, x_1) = 1$

$f(x_0, x_1, x_2) = \frac{1}{3}$ $\quad f(x_0, x_1, x_2, x_3) = 0$ $\quad f(x_0, x_1, x_2, x_3, x_4) = \frac{-1}{144}$

sub. all these values in Newton's divided diff. formula, we get

$$f(x) = 0 + (x-1).1 + (x-1)(x-2).\frac{1}{3} + (x-1)(x-2)(x-4).0 +$$
$$(x-1)(x-2)(x-4)(x-8).\frac{-1}{144}$$

$$f(x) = \frac{x^2-1}{3} - \frac{1}{144}(x^4 - 15x^3 + 70x^2 - 120x + 64)$$

At $x=3$, $f(3) = \frac{3^2-1}{3} - \frac{1}{144}(3^4 - 15.3^3 + 70.3^2 - 360 + 64)$.

$= 2.5972$.

NEWTONS DIVIDED DIFFERENCE FORMULA

1. Given the values.

x	5	7	11	13	17
$f(x)$	150	392	1452	2366	5202

Ans:- 810.

Evaluate $f(9)$, using Newton's divided difference formula.

2. Determine $f(x)$ as a polynomial in x for the following data.

x	-4	-1	0	2	5
$f(x)$	1245	33	5	9	1335

Ans:- $3x^4 - 5x^3 + 6x^2 - 14x + 5$

3. Find the third divided difference with arguments 2, 4, 9, 10 of the function $f(x) = x^3 - 2x$. Ans:- 1

4. Use Newton's divided difference method to compute $f(5.5)$ from the following data.

x	0	1	4	5	6
$f(x)$	1	14	15	6	3

Ans:- 3.09

5. Using Newtons divided difference formula Evaluate $f(8)$ and $f(15)$ given

x	4	5	7	10	11	13
$f(x)$	48	100	294	900	1210	2028

Ans:- 448, 3150.

6. obtain the Newton's divided difference interpolation polynomial and hence find $f(6)$.

x	3	7	9	10
$f(x)$	168	120	72	63

Ans:- 133.19

7. Using Newtons divided difference interpolation, find the polynomial of the given data.

x	-1	0	1	3
$f(x)$	2	1	0	-1

Ans:- $f(x) = \frac{1}{24}x^3 - \frac{7}{6}x^2 - 25x + \frac{557}{60}x + 1$

Symbolic Relations and separation of Symbols :-

Forward difference operator :- The forward difference operator is denoted by Δ and is defined as $\Delta f(x) = f(x+h) - f(x)$.

Where h is the increment in x.

Backward difference operator :- The backward difference operator is denoted by ∇ and is defined as $\nabla f(x) = f(x) - f(x-h)$.

Central difference operator :- The central difference operator is denoted by δ and is defined as

$$\delta y_{\frac{1}{2}} = y_1 - y_0$$

$$\delta y_{\frac{3}{2}} = y_2 - y_1$$

$$\delta y_{\frac{5}{2}} = y_3 - y_2$$

$$\cdots\cdots$$

$$\delta y_{\frac{2n-1}{2}} = y_n - y_{n-1}$$

Mean or Average operator :-

The mean or average operator μ is defined as $\mu y_r = \frac{1}{2}\left[y_{r+\frac{1}{2}} + y_{r-\frac{1}{2}}\right]$

The shift operator :-

The shift operator E is defined by the equation $Ey_r = y_{r+1}$.

This shows that the effect of E is to shift the functional value y_r to the next higher value. y_{r+1}.

A second operation with E gives $E^2 y_r = E(Ey_r) = E(y_{r+1}) = y_{r+2}$.

In general $E^n y_r = y_{r+n}$

We have $\quad \Delta f(x) = f(x+h) - f(x)$.

$$f(x+h) = f(x) + \Delta f(x) = (1+\Delta) f(x)$$

This shows that the operator $1+\Delta$ operating on $f(x)$ shifts $f(x)$ forward to its immediately succeeding value $f(x+h)$ (2)

We denote this operator by E and refer to it as the (first order) shift operator.

Thus by definition $E = 1+\Delta$

and we have $Ef(x) = f(x+h)$. $\left[\because Ef(x) = (1+\Delta)f(x)\right]$

$$E^2 f(x) = E\{Ef(x)\} = E\{f(x+h)\} = f(x+2h)$$

$$E^3 f(x) = E\{E^2 f(x)\} = E\{f(x+2h)\} = f(x+3h)$$

. . .

In general $E^n f(x)$ is defined by $E^n f(x) = f(x+nh)$, $n = 1, 2, 3 \ldots$ (1)

The operator E^n shifts the value of a function at x to its value at $x+nh$. This operator is referred to as the n^{th} order shift operator.

The formula (1) can be put in the following alternative form

$$E^n y_r = y_{r+n}$$

The inverse shift operator :-

Inverse operator E^{-1} is defined as $E^{-1} y_r = y_{r-1}$.

In general $E^{-n} y_r = y_{r-n}$.

Since $\Delta f(x) = \nabla f(x+h)$

We have $\nabla f(x) = \Delta f(x-h) = f(x) - f(x-h)$

$$f(x-h) = f(x) - \nabla f(x)$$

$$= (1-\nabla) f(x).$$

Thus, the operator $(1-\nabla)$ operating on $f(x)$ shifts $f(x)$ backward to its immediately preceding value $f(x-h)$. We denote this operator by E^{-1} (or $\frac{1}{E}$) and refer to it as the (first-order) inverse shift operator. Thus $E^{-1} = 1-\nabla$. (3)

and we have $E^{-1}f(x) = f(x-h)$.

$$E\left[E^{-1}f(x)\right] = E\,f(x-h) = f(x)$$

$$E^{-1}\left[E\,f(x)\right] = E^{-1}[f(x+h)] = f(x).$$

This means that the operator E^{-1} is actually the inverse of the operator E.

The nth order inverse shift operator E^{-n} is defined by an expression

$$E^{-n}f(x) = f(x-nh) \quad , \quad n = 1, 2, 3, \ldots \quad (1)$$

Thus, the operator E^{-n} shifts the value of a function at x to its value at $x-nh$.

The formula (1) can be put in the following alternative form.

$$E^{-n}y_r = y_{r-n}$$

<u>Relation between the Operators</u> :-

(i) $\Delta = E-1$

We have $\Delta y_0 = y_1 - y_0$.

$$\Delta y_0 = E^1 y_0 - y_0 \qquad \left[\because E^n y_r = y_{r+n}\right]$$

$$\Delta y_0 = (E-1)y_0$$

$$\Delta = E-1.$$

(OR)

$$\Delta f(x) = f(x+h) - f(x) \qquad \because E f(x) = f(x+h)$$

$$= E f(x) - f(x) = (E-1)f(x)$$

$$\therefore \Delta = E-1$$

(ii) $\nabla = 1 - E^{-1}$

We have. $\nabla y_1 = y_1 - y_0$

$$= y_1 - E^{-1} y_1 \qquad \left[\because E^{-n} y_r = y_{r-n}\right]$$

$$\nabla y_1 = (1 - E^{-1}) y_1$$

$$\nabla = (1 - E^{-1})$$

(OR)

$$\nabla f(x) = f(x) - f(x-h)$$

$$= f(x) - E^{-1} f(x) \qquad \because E^{-1} f(x) = f(x-h)$$

$$\nabla f(x) = (1 - E^{-1}) f(x)$$

$$\nabla = 1 - E^{-1}.$$

(iii) $\delta = E^{1/2} - E^{-1/2}$

We have $\delta y_{\frac{1}{2}} = y_1 - y_0$

$$= E^{1/2} y_{1/2} - E^{-1/2} y_{1/2} \qquad \left[\because E^{n} y_r = y_{r+n}, \quad E^{-n} y_r = y_{r-n}\right]$$

$$\delta y_{\frac{1}{2}} = (E^{1/2} - E^{-1/2}) y_{1/2}$$

$$\delta = E^{1/2} - E^{-1/2}$$

(iv) $\mu = \frac{1}{2}\left[E^{1/2} + E^{-1/2}\right]$

We have $\mu y_r = \frac{1}{2}\left[y_{r+\frac{1}{2}} + y_{r-\frac{1}{2}}\right]$

$$= \frac{1}{2}\left[E^{1/2} y_r + E^{-1/2} y_r\right]$$

$$\mu y_r = \frac{1}{2}\left[E^{1/2} + E^{-1/2}\right] y_r$$

$$\mu = \frac{1}{2}\left[E^{1/2} + E^{-1/2}\right].$$

Prove the following (a) $y_2 = y_0 + 2\Delta y_0 + \Delta^2 y_0$. (6)

(b) $y_3 = y_0 + 3\Delta y_0 + 3\Delta^2 y_0 + \Delta^3 y_0$.

Sol: (a) We have $\Delta y_0 = y_1 - y_0 \implies y_1 = y_0 + \Delta y_0$.

$$\Delta y_1 = y_2 - y_1 \implies y_2 = y_1 + \Delta y_1.$$

$$\Delta^2 y_0 = \Delta(\Delta y_0) = \Delta(y_1 - y_0) = \Delta y_1 - \Delta y_0$$

$$\Delta^2 y_0 = \Delta y_1 - \Delta y_0 \implies \Delta y_1 = \Delta y_0 + \Delta^2 y_0.$$

$$y_2 = y_1 + \Delta y_1$$

$$= (y_0 + \Delta y_0) + (\Delta y_0 + \Delta^2 y_0)$$

$$\therefore y_2 = y_0 + 2\Delta y_0 + \Delta^2 y_0$$

(b) $\Delta y_2 = y_3 - y_2 \implies y_3 = y_2 + \Delta y_2$.

$$\Delta^2 y_1 = \Delta(\Delta y_1) = \Delta(y_2 - y_1) = \Delta y_2 - \Delta y_1.$$

$$\Delta y_2 = \Delta y_1 + \Delta^2 y_1.$$

$$\Delta^3 y_0 = \Delta^2(\Delta y_0) = \Delta^2(y_1 - y_0) = \Delta^2 y_1 - \Delta^2 y_0$$

$$\Delta^2 y_1 = \Delta^2 y_0 + \Delta^3 y_0.$$

$$y_3 = y_2 + \Delta y_2.$$

$$= (y_0 + 2\Delta y_0 + \Delta^2 y_0) + (\Delta y_1 + \Delta^2 y_1)$$

$$= (y_0 + 2\Delta y_0 + \Delta^2 y_0) + (\Delta y_0 + \Delta^2 y_0) + (\Delta^2 y_0 + \Delta^3 y_0)$$

$$y_3 = y_0 + 3\Delta y_0 + 3\Delta^2 y_0 + \Delta^3 y_0.$$

(v) $\mu^2 = 1 + \frac{1}{4}\delta^2$

We have $\mu = \frac{1}{2}\left[E^{1/2} + E^{-1/2}\right]$

$$\mu^2 = \frac{1}{4}\left(E^{1/2} + E^{-1/2}\right)^2$$

$$= \frac{1}{4}\left[\left(E^{1/2} - E^{-1/2}\right)^2 + 4E^{1/2}E^{-1/2}\right]$$

$$= \frac{1}{4}\left[\delta^2 + 4\right]$$

$$\mu^2 = 1 + \frac{1}{4}\delta^2$$

(vi) $E = e^{hD}$

We have $Ef(x) = f(x+h)$.

By Taylor series

$$f(x+h) = f(x) + \frac{h}{1!}f'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \cdots$$

$$= f(x) + \frac{h}{1!}\frac{d}{dx}(f(x)) + \frac{h^2}{2!}\frac{d^2}{dx^2}(f(x)) + \frac{h^3}{3!}\frac{d^3}{dx^3}(f(x)) + \cdots$$

$$= f(x) + hD(f(x)) + \frac{h^2}{2!}D^2(f(x)) + \frac{h^3}{3!}D^3(f(x)) + \cdots$$

$$f(x+h) = \left(1 + hD + \frac{h^2D^2}{2!} + \frac{h^3D^3}{3!} + \cdots\right)f(x).$$

$$Ef(x) = e^{hD}f(x) \qquad \left[\because e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right].$$

$$\therefore \quad E = e^{hD}$$

Prove the following (7) (30)

(i) $\Delta y_r^2 = (y_r + y_{r+1})\,\Delta y_r$

(ii) $\sum_{r=0}^{n-1} \Delta^2 y_r = \Delta y_n - \Delta y_0$

Sol: (i) $\Delta y_r^2 = y_{r+1}^2 - y_r^2$

$$= (y_{r+1} + y_r)(y_{r+1} - y_r)$$

$$= (y_{r+1} + y_r)\,\Delta y_r .$$

(ii) $\sum_{r=0}^{n-1} \Delta^2 y_r = \Delta^2 y_0 + \Delta^2 y_1 + \Delta^2 y_2 + \dots + \Delta^2 y_{n-2} + \Delta^2 y_{n-1}$

$$= \Delta(\Delta y_0) + \Delta(\Delta y_1) + \Delta(\Delta y_2) + \dots + \Delta(\Delta y_{n-2}) + \Delta(\Delta y_{n-1})$$

$$= \Delta(y_1 - y_0) + \Delta(y_2 - y_1) + \Delta(y_3 - y_2) + \dots + \Delta(y_{n-1} - y_{n-2}) + \Delta(y_n - y_{n-1})$$

$$= (\Delta y_1 - \Delta y_0) + (\Delta y_2 - \Delta y_1) + (\Delta y_3 - \Delta y_2) + \dots + (\Delta y_{n-1} - \Delta y_{n-2}) + (\Delta y_n - \Delta y_{n-1})$$

$$= \Delta y_n - \Delta y_0$$

Evaluate (i) $\Delta \cos x$ (ii) $\Delta \log f(x)$ (iii) $\Delta^2 \sin(px+q)$ (iv) $\Delta \tan^{-1} x$.

(v) $\Delta^n (e^{ax+b})$.

Sol: We have $\Delta f(x) = f(x+h) - f(x)$.

(i) Let $f(x) = \cos x$

$$\Delta \cos x = \cos(x+h) - \cos x = -2\sin\left(x + \frac{h}{2}\right)\sin\left(\frac{h}{2}\right).$$

(ii) Let $f_1(x) = \log f(x)$.

$$\Delta \log f(x) = \log f(x+h) - \log f(x) = \log\left[\frac{f(x+h)}{f(x)}\right].$$

$$= \log\left[\frac{f(x) + \Delta f(x)}{f(x)}\right] = \log\left[1 + \frac{\Delta f(x)}{f(x)}\right].$$

(iii) Let $f(x) = \sin(px+q)$.

$$\Delta \sin(px+q) = \sin[p(x+h)+q] - \sin(px+q)$$

$$= 2\cos\left(px+q+\frac{ph}{2}\right)\sin\left(\frac{ph}{2}\right)$$

$$= 2\sin\left(\frac{ph}{2}\right)\sin\left(\frac{\pi}{2}+px+q+\frac{ph}{2}\right)$$

$$\Delta^2 \sin(px+q) = 2\sin\left(\frac{ph}{2}\right)\Delta\left[\sin\left[(px+q)+\frac{1}{2}(\pi+ph)\right]\right].$$

$$= 2\sin\left(\frac{ph}{2}\right)\left\{\sin\left[p(x+h)+q)+\frac{1}{2}(\pi+ph)\right] - \sin\left[(px+q)+\frac{1}{2}(\pi+ph)\right]\right\}$$

$$= 2\sin\left(\frac{ph}{2}\right)\ 2\cos\left[(px+q)+\left(\frac{ph}{2}\right)+\frac{1}{2}(\pi+ph)\right]\sin\left(\frac{ph}{2}\right)$$

$$= \left[2\sin\left(\frac{ph}{2}\right)\right]^2 \sin\left[(px+q)+2\cdot\frac{1}{2}(\pi+ph)\right].$$

(iv) Let $f(x) = \tan^{-1}x$.

$$\Delta \tan^{-1}x = \tan^{-1}(x+h) - \tan^{-1}x.$$

$$= \tan^{-1}\left(\frac{x+h-x}{1+x(x+h)}\right)$$

$$= \tan^{-1}\left(\frac{h}{1+x(x+h)}\right)$$

(v) Let $f(x) = e^{ax+b}$

$$\Delta e^{ax+b} = e^{a(x+h)+b} - e^{ax+b} = e^{ax+b}(e^{ah}-1)$$

$$\Delta^2 e^{ax+b} = \Delta[\Delta(e^{ax+b})] = \Delta\left[e^{ax+b}(e^{ah}-1)\right] = (e^{ah}-1)\,\Delta(e^{ax+b})$$

$$= (e^{ah}-1)(e^{ah}-1)e^{ax+b} = (e^{ah}-1)^2 e^{ax+b}.$$

Proceeding like this, we get $\Delta^n(e^{ax+b}) = (e^{ah}-1)^n e^{ax+b}$

If the interval of differencing is unity, prove that 9 31

$$\Delta \tan^{-1}\left(\frac{x-1}{x}\right) = \tan^{-1}\left(\frac{1}{2x^2}\right)$$

Sol: We have $\Delta f(x) = f(x+h) - f(x)$

Let $f(x) = \tan^{-1}\left(\frac{x-1}{x}\right)$

Given $h = 1$.

$$\Delta \tan^{-1}\left(\frac{x-1}{x}\right) = \Delta \tan^{-1}\left(1-\frac{1}{x}\right)$$

$$= \tan^{-1}\left(1-\frac{1}{x+1}\right) - \tan^{-1}\left(1-\frac{1}{x}\right)$$

$$= \tan^{-1}\left(\frac{\left(1-\frac{1}{x+1}\right)-\left(1-\frac{1}{x}\right)}{1+\left(1-\frac{1}{x+1}\right)\left(1-\frac{1}{x}\right)}\right)$$

$$= \tan^{-1}\left(\frac{\frac{1}{x}-\frac{1}{x+1}}{1+\left(\frac{x}{x+1}\right)\left(\frac{x-1}{x}\right)}\right)$$

$$= \tan^{-1}\left(\frac{\frac{x+1-x}{x(x+1)}}{\frac{(x+1)x+x(x-1)}{x(x+1)}}\right)$$

$$= \tan^{-1}\left(\frac{1}{2x^2}\right).$$

Evaluate $\Delta[f(x)g(x)]$.

Sol: We have $\Delta f(x) = f(x+h) - f(x)$.

$$\Delta[f(x)g(x)] = f(x+h)g(x+h) - f(x)g(x)$$

$$= f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)$$

$$= f(x+h)\left[g(x+h)-g(x)\right] + g(x)\left[f(x+h)-f(x)\right]$$

$$= f(x+h)\,\Delta g(x) + g(x)\,\Delta f(x).$$

Evaluate $\Delta\left[\frac{f(x)}{g(x)}\right]$

Sol: We have $\Delta f(x) = f(x+h) - f(x)$

$$\Delta\left[\frac{f(x)}{g(x)}\right] = \frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}$$

$$= \frac{f(x+h)\,g(x) - g(x+h)\,f(x)}{g(x+h)\,g(x)}$$

$$= \frac{f(x+h)\,g(x) - f(x)\,g(x) + g(x)\,f(x) - g(x+h)\,f(x)}{g(x+h)\,g(x)}$$

$$= \frac{g(x)\left[f(x+h) - f(x)\right] - f(x)\left[g(x+h) - g(x)\right]}{g(x+h)\,g(x)}$$

$$= \frac{g(x)\,\Delta f(x) - f(x)\,\Delta g(x)}{g(x+h)\,g(x)}.$$

Show that $\sum_{k=0}^{n-1} \Delta^2 f_k = \Delta f_n - \Delta f_0$.

Sol:- Let $y = f(x)$.

We know that the first forward difference is $\Delta y_k = y_{k+1} - y_k$.

Put $y_k = f(x_k) = f_k$, We get $\Delta f_k = f_{k+1} - f_k$.

The second difference is $\Delta^2 f_k = \Delta(\Delta f_k)$

$$= \Delta(f_{k+1} - f_k)$$

$$= \Delta f_{k+1} - \Delta f_k.$$

$$\sum_{k=0}^{n-1} \Delta^2 f_k = \sum_{k=0}^{n-1} (\Delta f_{k+1} - \Delta f_k)$$

$$= (\Delta f_1 - \Delta f_0) + (\Delta f_2 - \Delta f_1) + \cdots + (\Delta f_n - \Delta f_{n-1})$$

$$= \Delta f_n - \Delta f_0.$$

If $f(x) = e^{ax}$ show that $\Delta^n f(x) = (e^{ah}-1)^n e^{ax}$.

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Sol: Given that $f(x) = e^{ax}$.

We have $\Delta f(x) = f(x+h) - f(x)$

$= e^{a(x+h)} - e^{ax}$. Here h is the step size.

We have to show that $\Delta^n f(x) = (e^{ah}-1)^n e^{ax}$.

This can be proved by mathematical induction.

First we shall prove that this is true for $n=1$.

$(e^{ah}-1)^1 e^{ax} = e^{ah} \cdot e^{ax} - e^{ax}$.

$= e^{ah+ax} - e^{ax} = e^{a(x+h)} - e^{ax} = f(x+h) - f(x) = \Delta f(x)$.

$\therefore \Delta f(x_i) = f(x_i+h) - f(x_i)$.

Therefore, the result is true for $n=1$.

Assume that the problem is true for $n-1$.

Now consider, $\Delta^n f(x) = \Delta^{n-1}[\Delta f(x)]$

$= \Delta^{n-1}[f(x+h) - f(x)]$

$= \Delta^{n-1} f(x+h) - \Delta^{n-1} f(x)$

$= (e^{ah}-1)^{n-1} e^{a(x+h)} - (e^{ah}-1)^{n-1} e^{ax}$.

$= (e^{ah}-1)^{n-1}[e^{a(x+h)} - e^{ax}]$.

$= (e^{ah}-1)^{n-1}[e^{ax+ah} - e^{ax}]$

$= (e^{ah}-1)^{n-1}(e^{ax} e^{ah} - e^{ax})$

$= (e^{ah}-1)^{n-1}(e^{ah}-1) e^{ax}$.

$= (e^{ah}-1)^n \cdot e^{ax}$.

$\therefore \Delta^n f(x) = (e^{ah}-1)^n e^{ax}$.

Show that $\Delta\left(\frac{f_i}{g_i}\right) = \frac{g_i \Delta f_i - f_i \Delta g_i}{g_i\, g_{i+1}}$ 12

Sol:- We know that $\Delta f(x) = f(x+h) - f(x)$

$$\Delta\left(\frac{f_i}{g_i}\right) = \frac{f_{i+1}}{g_{i+1}} - \frac{f_i}{g_i}$$

$$\frac{g_i \Delta f_i - f_i \Delta g_i}{g_i\, g_{i+1}} = \frac{g_i(f_{i+1} - f_i) - f_i(g_{i+1} - g_i)}{g_i\, g_{i+1}}$$

$$= \frac{g_i f_{i+1} - f_i g_i - f_i g_{i+1} + f_i g_i}{g_i\, g_{i+1}} = \frac{g_i f_{i+1} - f_i g_{i+1}}{g_i\, g_{i+1}}$$

$$= \frac{f_{i+1}}{g_{i+1}} - \frac{f_i}{g_i}$$

$$\therefore \Delta\left(\frac{f_i}{g_i}\right) = \frac{g_i \Delta f_i - f_i \Delta g_i}{g_i\, g_{i+1}}$$

Show that $\Delta f_i^2 = (f_i + f_{i+1}) \Delta f_i$

Sol: We know that $\Delta f_k = f_{k+1} - f_k$

$$\Delta f_i^2 = f_{i+1}^2 - f_i^2$$

$$= (f_{i+1} - f_i)(f_{i+1} + f_i)$$

$$= (f_{i+1} + f_i)\Delta f_i$$

If the interval of differencing is unity prove that

$\Delta[x(x+1)(x+2)(x+3)] = 4(x+1)(x+2)(x+3)$

Sol:- Let $f(x) = x(x+1)(x+2)(x+3)$, We know that $\Delta f(x) = f(x+h) - f(x)$

$\Delta f(x) = f(x+1) - f(x)$ $[\because h = 1]$

$$= (x+1)(x+2)(x+3)(x+4) - x(x+1)(x+2)(x+3)$$

$$= 4(x+1)(x+2)(x+3)$$

Differences of a Polynomial :-

Result :- If $f(x)$ is a polynomial of degree n and the values of x are equally spaced then $\Delta^n f(x)$ is a constant.

proof :- Let $f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_{n-1} x + a_n$

Where $a_0, a_1, a_2, \ldots a_n$ are constants and $a_0 \neq 0$.

If h is the step length, We know that $\Delta f(x) = f(x+h) - f(x)$.

$$\Delta f(x) = \left[a_0 (x+h)^n + a_1 (x+h)^{n-1} + \cdots + a_{n-1}(x+h) + a_n\right]$$
$$- \left[a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n\right]$$

$$= a_0\left[\left\{n_{C_0} x^n + n_{C_1} x^{n-1} h + n_{C_2} x^{n-2} h^2 + \cdots\right\} - x^n\right]$$
$$+ a_1\left[\left\{(n-1)_{C_0} x^{n-1} + (n-1)_{C_1} x^{n-2} h + (n-1)_{C_2} x^{n-3} h^2 + \cdots\right\} - x^{n-1}\right] + \cdots + a_{n-1} h.$$

$$= a_0\left[\left\{ x^n + n x^{n-1} h + \frac{n(n-1)}{2!} x^{n-2} h^2 + \cdots\right\} - x^n\right]$$
$$+ a_1\left[\left\{ x^{n-1} + (n-1) x^{n-2} h + \frac{(n-1)(n-2)}{2!} x^{n-3} h^2 + \cdots\right\} - x^{n-1}\right] + \cdots + a_{n-1} h.$$

$$= a_0 n h x^{n-1} + b_2 x^{n-2} + b_3 x^{n-3} + \cdots + b_{n-3} x + b_{n-2}$$

Where $b_2, b_3, \ldots b_{n-2}$ are constants. Here this polynomial is of degree $(n-1)$.

Thus, the first difference of a polynomial of nth degree is a polynomial of degree $(n-1)$.

$$\Delta^2 f(x) = \Delta\left[\Delta f(x)\right]$$
$$= \Delta\left[a_0 n h x^{n-1} + b_2 x^{n-2} + b_3 x^{n-3} + \cdots + b_{n-1} x + b_{n-2}\right]$$

$$= \left[a_0 n h (x+h)^{n-1} + b_2 (x+h)^{n-2} + b_3 (x+h)^{n-3} + \cdots + b_{n-1}(x+h) + b_{n-2}\right]$$
$$- \left[a_0 n h x^{n-1} + b_2 x^{n-2} + b_3 x^{n-3} + \cdots + b_{n-1} x + b_{n-2}\right]$$

$$= a_0 n h \left[(x+h)^{n-1} - x^{n-1}\right] + b_2\left[(x+h)^{n-2} - x^{n-2}\right] + \cdots + b_{n-1}\left[(x+h) - x\right]$$

$$= a_0 n(n-1) h^2 x^{n-2} + c_3 x^{n-3} + \cdots + c_{n-4} x + c_{n-3}$$

Where $c_3, c_4, \ldots c_{n-3}$ are constants. This polynomial is of degree $(n-2)$

Thus, the second difference of a polynomial of degree n is a polynomial of degree $(n-2)$. Continuing like this we get, $\Delta^n f(x)$.

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$\Delta^n f(x) = a_0\, n(n-1)(n-2)\ldots 2.1 h^n = a_0 h^n (n!)$ Which is a constant.

Hence the result.

Note:- (i) As $\Delta^n f(x)$ is a constant, it follows that $\Delta^{n+1} f(x) = 0$;

$\Delta^{n+2} f(x) = 0, \ldots$

(ii) The converse of above result is also true. That is, if $\Delta^n f(x)$ is tabulated at equally spaced intervals and is a constant, then the function $f(x)$ is a polynomial of degree n.

If the values of x are specified with step length h, evaluate

(i) $\Delta^3\left[(1-x)(1-2x)(1+3x)\right]$ (ii) $\Delta^{10}\left[(1-x)(1-2x^2)(1-3x^3)(1-4x^4)\right]$.

Sol:- (i) Let $f(x) = (1-x)(1-2x)(1+3x)$

Which is a polynomial of degree 3 with 6 as the coefficient of x^3.

If $f(x)$ is a polynomial of degree n, a_0 is coefficient of x^n Then

$\Delta^n f(x) = a_0 h^n n!$ Where h is the step length.

$$\therefore \Delta^3 f(x) = \Delta^3\left[(1-x)(1-2x)(1+3x)\right] = 6h^3(3!) = 36h^3.$$

(ii) Let $f(x) = (1-x)(1-2x^2)(1-3x^3)(1-4x^4)$

Which is a polynomial of degree 10 with $(-1)(-2)(-3)(-4) = 24$ as the coefficient of x^{10}.

$$\therefore \Delta^{10} f(x) = \Delta^{10}\left[(1-x)(1-2x^2)(1-3x^3)(1-4x^4)\right] = 24\, h^{10}(10!).$$

Find the second difference of the polynomial $f(x) = x^4 - 12x^3 + 42x^2 - 30x + 9$ with interval of differencing $h = 2$.

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Sol: Let $f(x) = x^4 - 12x^3 + 42x^2 - 30x + 9$.

We know that $\Delta f(x) = f(x+h) - f(x)$

$$\Delta f(x) = f(x+2) - f(x) \qquad \because h = 2$$

$$\Delta f(x) = [(x+2)^4 - 12(x+2)^3 + 42(x+2)^2 - 30(x+2) + 9] - [x^4 - 12x^3 + 42x^2 - 30x + 9]$$

$$= 8x^3 - 48x^2 + 56x + 28.$$

second difference $\Delta^2 f(x) = \Delta[\Delta f(x)]$

$$= \Delta[8x^3 - 48x^2 + 56x + 28]$$

$$= [8(x+2)^3 - 48(x+2)^2 + 56(x+2) + 28] - [8x^3 - 48x^2 + 56x + 28]$$

$$= 48x^2 - 96x - 16.$$

If the interval of differencing is unity, prove that $\Delta\left[\frac{1}{f(x)}\right] = \frac{-\Delta f(x)}{f(x) f(x+1)}$

Sol: We know that $\Delta f(x) = f(x+h) - f(x)$

$$\Delta f(x) = f(x+1) - f(x) \qquad \because h = 1.$$

$$\Delta\left[\frac{1}{f(x)}\right] = \frac{1}{f(x+1)} - \frac{1}{f(x)}$$

$$= \frac{f(x) - f(x+1)}{f(x+1) f(x)}$$

$$= \frac{-[f(x+1) - f(x)]}{f(x+1) f(x)}$$

$$\Delta\left[\frac{1}{f(x)}\right] = \frac{-\Delta f(x)}{f(x+1) f(x)}$$

If the interval of differencing is unity, proove that $\Delta\left(\frac{2^x}{x!}\right) = \frac{2^x(1-x)}{(x+1)!}$

(16)

Sol: Let $f(x) = \frac{2^x}{x!}$

We have $\Delta f(x) = f(x+h) - f(x)$

$$\Delta f(x) = f(x+1) - f(x)$$

$$= \frac{2^{x+1}}{(x+1)!} - \frac{2^x}{x!}$$

$$= \frac{2^x \cdot 2}{(x+1)\, x!} - \frac{2^x}{x!} = \frac{2^x}{x!}\left(\frac{2}{x+1} - 1\right)$$

$$= \frac{2^x}{x!}\left(\frac{2-x-1}{(x+1)}\right)$$

$$= \frac{2^x(1-x)}{(x+1)!}$$

Proove that $\Delta \log f(x) = \log\left[1 + \frac{\Delta f(x)}{f(x)}\right]$

Sol:- We know that $\Delta f(x) = f(x+h) - f(x)$

$$\Delta \log f(x) = \log f(x+h) - \log f(x)$$

$$= \log \frac{f(x+h)}{f(x)} = \log\left[\frac{f(x) + \Delta f(x)}{f(x)}\right]$$

$$= \log\left[1 + \frac{\Delta f(x)}{f(x)}\right].$$

Evaluate $\Delta(x + \cos x)$

Sol:- Here $f(x) = x + \cos x$

We have $\Delta f(x) = f(x+h) - f(x)$

$$\Delta f(x) = \Delta(x + \cos x) = \left[(x+h) + \cos(x+h)\right] - \left[x + \cos x\right]$$

$$= h + \cos(x+h) - \cos x.$$

Prove that $E\nabla = \Delta = \nabla E$

Sol:- We know that $\Delta f(x) = f(x+h) - f(x)$, $\nabla f(x) = f(x) - f(x-h)$

$$(E\nabla)f(x) = E(\nabla f(x)) = E[f(x) - f(x-h)] \qquad \because Ef(x) = f(x+h)$$

$$E^{-1}f(x) = f(x-h)$$

$$= Ef(x) - Ef(x-h)$$

$$= f(x+h) - f(x)$$

$$= \Delta f(x)$$

$$E\nabla = \Delta.$$

$$(\nabla E)f(x) = \nabla(Ef(x)) = \nabla(f(x+h))$$

$$= f(x+h) - f(x)$$

$$= \Delta f(x)$$

$$\nabla E = \Delta.$$

$$\therefore \quad E\nabla = \Delta = \nabla E$$

Prove that $\delta E^{1/2} = \Delta$.

$$\delta u_{x+\frac{h}{2}} = (E^{1/2} - E^{-1/2})u_{x+\frac{h}{2}}$$

$$= E^{1/2}u_{x+\frac{h}{2}} - E^{-1/2}u_{x+\frac{h}{2}}$$

$$= u_{x+h} - u_x$$

$$\delta u_{x+\frac{h}{2}} = \Delta u_x.$$

$$\delta E^{1/2}u_x = \Delta u_x$$

$$\delta E^{1/2} = \Delta.$$

Prove that $hD = \log(1+\Delta) = -\log(1-\Delta) = \sinh^{-1}(\mu\delta)$.

Sol: We know that $e^{hD} = E = 1+\Delta$.

Taking logarithm both sides, we get

$$\log_e e^{hD} = \log(1+\Delta)$$

$$hD \log_e e = \log(1+\Delta)$$

$$hD = \log(1+\Delta)$$

We have $\nabla = 1 - E^{-1} \Rightarrow E^{-1} = 1 - \nabla$.

$$e^{-hD} = 1 - \nabla \qquad \because e^{hD} = E$$

Taking logarithm bothsides, we get

$$\log_e e^{-hD} = \log_e(1-\nabla)$$

$$-hD \log_e e = \log_e(1-\nabla)$$

$$hD = -\log_e(1-\nabla)$$

$$\sinh hD = \frac{e^{hD} - e^{-hD}}{2} = \frac{E - E^{-1}}{2} = \left(\frac{E^{1/2} + E^{-1/2}}{2}\right)\left(E^{1/2} - E^{-1/2}\right) = \mu\delta$$

$$\sinh hD = \mu\delta$$

$$hD = \sin^{-1}(\mu\delta).$$

Prove that $1 + \mu^2\delta^2 = \left(1 + \frac{1}{2}\delta^2\right)^2$

Sol:
$$1 + \mu^2\delta^2 = 1 + \left(\frac{E^{1/2} + E^{-1/2}}{2}\right)^2 \left(E^{1/2} - E^{-1/2}\right)^2$$

$$= 1 + \frac{\left[\left(E^{1/2} + E^{-1/2}\right)\left(E^{1/2} - E^{-1/2}\right)\right]^2}{4}$$

$$= 1 + \frac{(E - E^{-1})^2}{4}$$

$$= \frac{4 + (E - E^{-1})^2}{4} = \left(\frac{E + E^{-1}}{2}\right)^2 \text{ --- (1)}$$

$$\left(1 + \frac{1}{2}\delta^2\right)^2 = \left[1 + \frac{1}{2}\left(E^{1/2} - E^{-1/2}\right)^2\right]^2 = \left[1 + \frac{1}{2}\left(E + E^{-1} - 2\right)\right]^2$$

$$= \left(\frac{E + E^{-1}}{2}\right)^2 \text{ --- (2)}$$

From (1) and (2),

$$1 + \mu^2\delta^2 = \left(1 + \frac{1}{2}\delta^2\right)^2.$$

(19) (36)

Prove that $E^{1/2} = \mu + \frac{1}{2}\delta$.

Sol: $\mu + \frac{1}{2}\delta = \frac{E^{1/2} + E^{-1/2}}{2} + \frac{1}{2}(E^{1/2} - E^{-1/2})$ $\quad\because \mu = \frac{E^{1/2}+E^{-1/2}}{2}$

$\delta = E^{1/2} - E^{-1/2}$

$\mu + \frac{1}{2}\delta = E^{1/2}$

Prove that $E^{-1/2} = \mu - \frac{1}{2}\delta$.

Sol: We know that $\mu = \frac{E^{1/2} + E^{-1/2}}{2}$, $\delta = E^{1/2} - E^{-1/2}$

$$\mu - \frac{1}{2}\delta = \frac{E^{1/2} + E^{-1/2}}{2} - \frac{1}{2}(E^{1/2} - E^{-1/2})$$

$$= E^{-1/2}$$

Prove that $\mu\delta = \frac{1}{2}\Delta E^{-1} + \frac{1}{2}\Delta$.

Sol: We have $\mu = \frac{E^{1/2} + E^{-1/2}}{2}$, $\delta = E^{1/2} - E^{-1/2}$, $\Delta = E - 1$.

$$\frac{1}{2}\Delta E^{-1} + \frac{1}{2}\Delta = \frac{1}{2}\Delta(E^{-1}+1) = \frac{1}{2}(E-1)(E^{-1}+1)$$

$$= \frac{1}{2}(E - E^{-1}) = \mu\delta.$$

Prove that $\Delta = \frac{1}{2}\delta^2 + \delta\sqrt{1+\frac{\delta^2}{4}}$

Sol: We have $\Delta = E - 1$, $\delta = E^{1/2} - E^{-1/2}$

$$\frac{1}{2}\delta^2 + \delta\sqrt{1+\frac{\delta^2}{4}} = \frac{1}{2}\delta\left[\delta + 2\sqrt{1+\frac{\delta^2}{4}}\right]$$

$$= \frac{1}{2}\delta\left[\delta + \sqrt{4+\delta^2}\right]$$

$$= \frac{1}{2}\delta\left[(E^{1/2} - E^{-1/2}) + \sqrt{4 + (E^{1/2} - E^{-1/2})^2}\right]$$

$$= \frac{1}{2}\delta\left[E^{1/2} - E^{-1/2} + \sqrt{(E^{1/2} + E^{-1/2})^2}\right]$$

$$= \frac{1}{2}\delta\left[E^{1/2} - E^{-1/2} + E^{1/2} + E^{-1/2}\right]$$

$$= \delta E^{1/2} = (E^{1/2} - E^{-1/2})E^{1/2}$$

$$= E - 1$$

$$= \Delta.$$

Prove that $\nabla\Delta = \Delta - \nabla = \delta^2$.

Sol: $\Delta\nabla = (1-E^{-1})(E-1)$

$= E + E^{-1} - 2$

$= (E^{1/2} - E^{-1/2})^2$

$= \delta^2$

$\left[\because\ \nabla = 1 - E^{-1}\right.$

$\Delta = E - 1$

$\delta = E^{1/2} - E^{-1/2}$

$\Delta - \nabla = (E-1) - (1-E^{-1})$

$= E + E^{-1} - 2$

$= (E^{1/2} - E^{-1/2})^2$

$= \delta^2$

Prove that $(1+\Delta)(1-\nabla) = 1$.

Sol: We have $\Delta = E-1$, $\nabla = 1 - E^{-1}$.

$(1+\Delta)(1-\nabla) = E(1-\nabla)$

$= E(1-(1-E^{-1}))$

$= EE^{-1}$

$= 1.$

Prove that $\mu\delta = \frac{1}{2}(\Delta + \nabla)$.

Sol We have $\mu = \frac{1}{2}(E^{1/2} + E^{-1/2})$, $\delta = E^{1/2} - E^{-1/2}$, $\Delta = E-1$, $\nabla = 1-E^{-1}$

$\frac{1}{2}(\Delta + \nabla) = \frac{1}{2}(E - 1 + 1 - E^{-1})$

$= \frac{1}{2}(E - E^{-1})$

$= \frac{1}{2}(E^{1/2} + E^{-1/2})(E^{1/2} - E^{-1/2})$

$= \mu\delta$.

Prove that (i) $y_{n-2} = y_n - 2\nabla y_n + \nabla^2 y_n$ (ii) $y_{n-3} = y_n - 3\nabla y_n + 3\nabla^2 y_n - \nabla^3 y_n$.

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Sol:- By definition $\nabla y_n = y_n - y_{n-1}$.

$$\nabla y_{n-1} = y_{n-1} - y_{n-2}$$

$$\nabla^2 y_n = \nabla(\nabla y_n)$$

$$= \nabla(y_n - y_{n-1})$$

$$= \nabla y_n - \nabla y_{n-1}$$

$$= (y_n - y_{n-1}) - (y_{n-1} - y_{n-2})$$

$$\nabla^2 y_n = y_n - 2y_{n-1} + y_{n-2}$$

$$y_{n-2} = \nabla^2 y_n - y_n + 2y_{n-1}$$

$$= \nabla^2 y_n - y_n + y_{n-1} + y_{n-1} + y_n - y_n$$

$$= \nabla^2 y_n - (y_n - y_{n-1}) - (y_n - y_{n-1}) + y_n$$

$$= \nabla^2 y_n - \nabla y_n - \nabla y_n + y_n$$

$$y_{n-2} = y_n - 2\nabla y_n + \nabla^2 y_n.$$

(ii) We have $\nabla y_{n-2} = y_{n-2} - y_{n-3}$ (OR) $y_{n-3} = y_{n-2} - \nabla y_{n-2}$

$$y_{n-3} = y_{n-2} - \nabla y_{n-2}$$

$$= (y_n - 2\nabla y_n + \nabla^2 y_n) - \nabla(y_n - 2\nabla y_n + \nabla^2 y_n)$$

$$= y_n - 2\nabla y_n + \nabla^2 y_n - \nabla y_n + 2\nabla^2 y_n - \nabla^3 y_n$$

$$= y_n - 3\nabla y_n + 3\nabla^2 y_n - \nabla^3 y_n.$$

If h is the step length, prove that $\Delta[f(x-h).\Delta g(x-h)] = \nabla[f(x).\Delta g(x)]$

Sol:- $\Delta[f(x-h).\Delta g(x-h)] = \Delta[f(x-h)\{g(x) - g(x-h)\}]$

$$= \Delta\{f(x-h)g(x)\} - \Delta\{f(x-h)g(x-h)\}$$

$$= f(x-h)\Delta g(x) + g(x+h)\Delta f(x-h)$$

$$- f(x-h)\Delta g(x-h) - g(x)\Delta f(x-h)$$

$$= f(x-h)\left[g(x+h)-g(x)\right] + g(x+h)\left[f(x)-f(x-h)\right]$$

$$- f(x-h)\left[g(x)-g(x-h)\right] - g(x)\left[f(x)-f(x-h)\right]$$

$$= f(x)\left[g(x+h)-g(x)\right] + f(x-h)\left[g(x-h)-g(x)\right]$$

$$= f(x)\,\Delta g(x) + f(x-h)\,\Delta g(x-h).$$

$$= \nabla\left[f(x).\,\Delta g(x)\right].$$

Since $\nabla\left[f(x)\,\Delta g(x)\right] = \nabla\left[f(x).\left(g(x+h)-g(x)\right)\right]$

$$= \nabla\left[f(x)\,g(x+h) - f(x)g(x)\right]$$

$$= \nabla\left(f(x)\,g(x+h)\right) - \nabla\left(f(x)\,g(x)\right)$$

$$= \left[f(x)\,g(x+h) - f(x-h)\,g(x)\right] - \left[f(x)g(x) - f(x-h).g(x-h)\right]$$

$$= f(x)\left[g(x+h)-g(x)\right] + f(x-h)\left[g(x-h)-g(x)\right]$$

$$\nabla\left[f(x).\,\Delta g(x)\right] = f(x)\,\Delta g(x) + f(x-h)\,\Delta g(x-h)$$

If the values of x are equally spaced, prove that

$(\Delta - \nabla)f(x) = \Delta\nabla f(x)$

Sol. We have $\Delta f(x) = f(x+h) - f(x)$, $\nabla f(x) = f(x) - f(x-h)$.

$$\Delta\nabla f(x) = \Delta\left[\nabla f(x)\right]$$

$$= \Delta\left[f(x) - f(x-h)\right]$$

$$= \Delta f(x) - \Delta f(x-h)$$

$$= \Delta f(x) - \left[f(x) - f(x-h)\right]$$

$$= \Delta f(x) - \nabla f(x)$$

$$= (\Delta - \nabla)\,f(x)$$

$$\Delta\nabla = \Delta - \nabla.$$

Proove that $\Delta^2 = E^2 - 2E + 1$.

sol:- We know that $\Delta f(x) = f(x+h) - f(x)$, $E^n f(x) = f(x+nh)$

$$\Delta^2 f(x) = \Delta[\Delta f(x)] = \Delta[f(x+h) - f(x)]$$

$$= \Delta f(x+h) - \Delta f(x)$$

$$= [f(x+2h) - f(x+h)] - [f(x+h) - f(x)].$$

$$= f(x+2h) - 2f(x+h) + f(x)$$

$$= E^2 f(x) - 2Ef(x) + f(x)$$

$$\Delta^2 f(x) = (E^2 - 2E + 1) f(x)$$

$$\Delta^2 = E^2 - 2E + 1.$$

Proove that $\Delta E = E\Delta$

sol:- We know that $\Delta f(x) = f(x+h) - f(x)$, $E^n f(x) = f(x+nh)$.

$$\Delta E f(x) = \Delta[Ef(x)]$$

$$= \Delta f(x+h)$$

$$= f(x+2h) - f(x+h)$$

$$E\Delta f(x) = E[\Delta f(x)]$$

$$= E[f(x+h) - f(x)]$$

$$= Ef(x+h) - Ef(x)$$

$$= f(x+2h) - f(x+h)$$

$$\therefore \Delta E f(x) = E\Delta f(x)$$

$$\Delta E = E\Delta.$$

Proove that $E^{-1}\Delta = \Delta E^{-1} = \nabla$.

sol:- We have $\Delta f(x) = f(x+h) - f(x)$, $E^n f(x) = f(x+nh)$, $\nabla f(x) = f(x) - f(x-h)$

$$\Delta E^{-1} f(x) = \Delta[E^{-1} f(x)] = \Delta f(x-h) = f(x) - f(x-h) = \nabla f(x)$$

$$\Delta E^{-1} f(x) = \nabla f(x)$$

$$\Delta E^{-1} = \nabla.$$

$$E^{-1}\Delta f(x) = E^{-1}[\Delta f(x)] = E^{-1}[f(x+h) - f(x)]$$
$$= E^{-1}f(x+h) - E^{-1}f(x)$$
$$= f(x) - f(x-h)$$
$$= \nabla f(x)$$
$$E^{-1}\Delta f(x) = \nabla f(x)$$
$$E^{-1}\Delta = \nabla.$$

Find (i) $\left(\frac{\Delta^2}{E}\right)f(x)$ (ii) $\frac{\Delta^2 f(x)}{Ef(x)}$ Deduce that $\left\{\left(\frac{\Delta^2}{E}\right)e^x\right\}\left\{\frac{Ee^x}{\Delta^2 e^x}\right\} = e^x.$

Sol: We know that $\Delta f(x) = f(x+h) - f(x)$, $E^n f(x) = f(x+nh)$

$$\Delta = E - 1$$

$$\left(\frac{\Delta^2}{E}\right)f(x) = \frac{(E-1)^2}{E}f(x) = E^{-1}(E-1)^2 f(x) = E^{-1}(E^2+1-2E)f(x)$$
$$= (E + E^{-1} - 2)f(x)$$
$$= Ef(x) + E^{-1}f(x) - 2f(x)$$
$$= f(x+h) + f(x-h) - 2f(x)$$

Find $\frac{\Delta^2 f(x)}{Ef(x)}$

Sol: We know that $\Delta f(x) = f(x+h) - f(x)$, $E^n f(x) = f(x+nh)$

$$\frac{\Delta^2 f(x)}{Ef(x)} = \frac{(E-1)^2 f(x)}{Ef(x)} = \frac{(E^2-2E+1)f(x)}{Ef(x)} = \frac{f(x+2h) - 2f(x+h) + f(x)}{f(x+h)}$$

Now take $f(x) = e^x$ in the above results, we get.

$$\left(\frac{\Delta^2}{E}\right)e^x = e^{x+h} - 2e^x + e^{x-h}$$

$$\frac{\Delta^2 e^x}{Ee^x} = \frac{e^{x+2h} - 2e^{x+h} + e^x}{e^{x+h}}.$$

$$\left\{\left(\frac{\Delta^2}{E}\right)e^x\right\}\left\{\frac{Ee^x}{\Delta^2 e^x}\right\} = \frac{(e^{x+h} - 2e^x + e^{x-h})(e^{x+h})}{e^{x+2h} - 2e^{x+h} + e^x} = \frac{e^x(e^{x+2h} - 2e^{x+h} + e^x)}{e^{x+2h} - 2e^{x+h} + e^x}$$

$$= e^x.$$

Prove that $\Delta^3 y_2 = \nabla^3 y_5$ (25)

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Sol: We have $\Delta = E-1, \quad \nabla = 1-E^{-1}$

$$\Delta^3 y_2 = (E-1)^3 y_2 = (E^3-3E^2+3E-1)y_2$$
$$= E^3y_2 - 3E^2y_2 + 3Ey_2 - y_2$$
$$= y_5 - 3y_4 + 3y_3 - y_2$$
$$\nabla^3 y_5 = (1-E^{-1})^3 y_5 = (1-3E^{-1}+3E^{-2}-E^{-3})y_5$$
$$= y_5 - 3E^{-1}y_5 + 3E^{-2}y_5 - E^{-3}y_5$$
$$= y_5 - 3y_4 + 3y_3 - y_2$$
$$\therefore \Delta^3 y_2 = \nabla^3 y_5.$$

Prove that $(E^{1/2}+E^{-1/2})(1+\Delta)^{1/2} = 2+\Delta$

Sol:- We have $\Delta = E-1$.

$$(E^{1/2}+E^{-1/2})(1+\Delta)^{1/2} = (E^{1/2}+E^{-1/2})E^{1/2}$$
$$= E+1$$
$$= E+1+1-1$$
$$= 2+E-1$$
$$= 2+\Delta.$$

If y_x is a polynomial for which fifth difference is constant and $y_1+y_7 = -7845$, $y_2+y_6 = 686$, $y_3+y_5 = 1088$ find y_4.

Sol:- Starting with y_1 instead of y_0, we note that $\Delta^6 y_1 = 0$.

We have $\Delta = E-1$.

$$\Delta^6 y_1 = 0 \Rightarrow (E-1)^6 y_1 = 0 \Rightarrow (E^6-6E^5+15E^4-20E^3+15E^2-6E+1)y_1 = 0$$
$$E^6y_1 - 6E^5y_1 + 15E^4y_1 - 20E^3y_1 + 15E^2y_1 - 6Ey_1 + y_1 = 0.$$
$$y_7 - 6y_6 + 15y_5 - 20y_4 + 15y_3 - 6y_2 + y_1 = 0.$$
$$(y_1+y_7) - 6(y_2+y_6) + 15(y_3+y_5) - 20y_4 = 0.$$
$$y_4 = \frac{1}{20}\left[(y_1+y_7) - 6(y_2+y_6) + 15(y_3+y_5)\right]$$
$$= \frac{1}{20}\left[-784 - 6(686) + 15(1088)\right] = 571.$$

Given $y_0 = 3$, $y_1 = 12$, $y_2 = 81$, $y_3 = 200$, $y_4 = 100$ and $y_5 = 8$ find $\Delta^5 y_0$.

sol: We know that $E = 1 + \Delta$

$$\Delta^5 y_0 = (E-1)^5 y_0 = (E^5 - 5c_1 E^4 + 5c_2 E^3 - 5c_3 E^2 + 5c_4 E - 1)\, y_0$$

$$= (E^5 - 5E^4 + 10E^3 - 10E^2 + 5E - 1)\, y_0$$

$$= E^5 y_0 - 5E^4 y_0 + 10E^3 y_0 - 10E^2 y_0 + 5E y_0 - y_0$$

$$= y_5 - 5y_4 + 10y_3 - 10y_2 + 5y_1 - y_0$$

$$= 8 - 500 + 2000 - 810 + 60 - 3$$

$$= 755$$

If $y_0 = 5$, $y_1 = 11$, $y_2 = 22$, $y_3 = 40$, $y_5 = 140$ find y_4 given that the general term is represented by a fourth degree polynomial.

sol:- We know that $E = 1 + \Delta$.

Since y_n is represented by a 4^{th} degree polynomial, we have $\Delta^5 y_n = 0$.

i.e $(E-1)^5 y_n = 0$.

$$(5c_0 E^5 - 5c_1 E^4 + 5c_2 E^3 - 5c_3 E^2 + 5c_4 E - 1)\, y_n = 0.$$

$$(E^5 - 5E^4 + 10E^3 - 10E^2 + 5E - 1)\, y_n = 0$$

$$E^5 y_n - 5E^4 y_n + 10E^3 y_n - 10E^2 y_n + 5E y_n - y_n = 0.$$

Take $n = 0$

$$E^5 y_0 - 5E^4 y_0 + 10E^3 y_0 - 10E^2 y_0 + 5E y_0 - y_0 = 0.$$

$$y_5 - 5y_4 + 10y_3 - 10y_2 + 5y_1 - y_0 = 0.$$

$$140 - 5y_4 + 400 - 220 + 55 - 5 = 0$$

$$5y_4 = 370$$

$$y_4 = 74.$$

Prove that $\Delta + \nabla = \frac{\Delta}{\nabla} - \frac{\nabla}{\Delta}$ (27)

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sol:- $\left(\frac{\Delta}{\nabla} - \frac{\nabla}{\Delta}\right) f(x) = \left(\Delta \nabla^{-1} - \nabla \Delta^{-1}\right) f(x)$

$= \left(\Delta(1-E^{-1})^{-1} - \nabla(E-1)^{-1}\right) f(x)$

$= \left(\Delta\left(\frac{E-1}{E}\right)^{-1} - \nabla(E-1)^{-1}\right) f(x)$

$= \left(\Delta\left(\frac{E}{E-1}\right) - \frac{\nabla}{E-1}\right) f(x)$

$= \frac{1}{E-1}\left(\Delta E - \nabla\right) f(x)$

$= \frac{1}{E-1}\left((E-1)E - \left(1-\frac{1}{E}\right)\right) f(x)$

$= \frac{1}{E-1}\left((E-1)E - \left(\frac{E-1}{E}\right)\right) f(x)$

$= \left(E - \frac{1}{E}\right) f(x)$ $\left[\because \Delta = E-1,\ \nabla = 1-E^{-1}\right.$

$= (\Delta + \nabla) f(x)$ $\left.\Delta + \nabla = E - E^{-1}.\right]$

$\therefore \frac{\Delta}{\nabla} - \frac{\nabla}{\Delta} = \Delta + \nabla$

Evaluate $(E^{-1}\Delta)x^3$ taking $h = 1$.

sol:- Given that $f(x) = x^3,\ h = 1$

We know that $\Delta f(x) = f(x+h) - f(x)$

$\Delta f(x) = f(x+1) - f(x)$

$\Delta x^3 = (x+1)^3 - x^3$

$\Delta x^3 = x^3 + 1 + 3x^2 + 3x - \cancel{x^3}$

$\Delta x^3 = 3x^2 + 3x + 1.$

$E^{-1}(\Delta x^3) = E^{-1}(3x^2 + 3x + 1)$

We know that $E^{-1}f(x) = f(x-h)$.

$= 3(x-1)^2 + 3(x-1) + 1$

$= 3x^2 + 3x + 1.$

If $h=1$ is the step length, prove that

$$\Delta^n u_{x-n} = u_x - n u_{x-1} + \frac{n(n-1)}{2!} u_{x-2} + \cdots + (-1)^n u_{x-n}.$$

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Sol: Writing $u_x = f(x)$, we get

$$u_x - n u_{x-1} + \frac{n(n-1)}{2!} u_{x-2} + \cdots + (-1)^n u_{x-n}$$

$$= f(x) - n f(x-1) + \frac{n(n-1)}{2!} f(x-2) + \cdots + (-1)^n f(x-n).$$

We know that $E^{-n} f(x) = f(x-nh)$.

$$= f(x) - n E^{-1} f(x) + \frac{n(n-1)}{2!} E^{-2} f(x) + \cdots + (-1)^n E^{-n} f(x).$$

$$= \left\{1 - n E^{-1} + \frac{n(n-1)}{2!} E^{-2} + \cdots + (-1)^n E^{-n}\right\} f(x)$$

$$= (1 - E^{-1})^n f(x).$$

$$= \left(1 - \frac{1}{E}\right)^n f(x) = \left(\frac{E-1}{E}\right)^n f(x)$$

$$= \frac{\Delta^n}{E^n} f(x) = \Delta^n E^{-n} f(x)$$

$$= \Delta^n (E^{-n} f(x))$$

$$= \Delta^n f(x-n) \qquad (\because E^{-n} f(x) = f(x-nh),\ h=1)$$

$$= \Delta^n u_{x-n}.$$

Differential Equation :- An equation involving differentials of one
dependent variable and its derivatives with respect to one or more
independent variables is called a differential equation.

Ordinary Differential Equation :- A differential equation is said
to be ordinary if the derivatives in the equation have reference
to only one a single independent variable.

Eg:- $\frac{dy}{dx} = x+y$, $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = x^2$.

Initial and Boundary value Problems :-

An ordinary differential equation of n^{th} order is of the form
$F(x, y, y', y'', y''', \ldots y^{(n)}) = 0$.
Its general solution will contain n arbitrary constants and it will be
of the form $f(x, y, c_1, c_2, c_3 \ldots c_n) = 0$.
To obtain its particular solution, n conditions must be given so that
the constants $c_1, c_2, c_3 \ldots c_n$ can be determined.
Problems in which $y, y', y'', \ldots y^{(n-1)}$ are all specified at the same value
of x say x_0 are called initial value problems.
If the conditions on y are prescribed at n distinct points then the
problems are called boundary value problems.
Problems in which function is prescribed at k different points and
(n-k) derivatives are prescribed at the same point are called mixed
value problems.

Taylor series Method :-

Consider the initial value problem $y' = \frac{dy}{dx} = f(x,y)$, subject to $y = y_0$

when $x = x_0$.

$y(x)$ can be expanded about the point x_0 in a Taylor series in powers

of $(x - x_0)$

$$y(x) = y(x_0) + \frac{(x-x_0)}{1!} \cdot y'(x_0) + \frac{(x-x_0)^2}{2!} y''(x_0) + \cdots + \frac{(x-x_0)^n}{n!} y^{(n)}(x_0) + \cdots$$

Where $y^i(x_0)$ is the ith derivative of $y(x)$ at $x = x_0$.

Let $x - x_0 = h$ (i.e $x = x_1 = x_0 + h$) we can write the Taylor's series as

$$y(x) = y(x_0) + \frac{h}{1!} y'(x_0) + \frac{h^2}{2!} y''(x_0) + \frac{h^3}{3!} y'''(x_0) + \cdots$$

$$y_1 = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \cdots$$

Substituting the values of $y_0, y_0', y_0'' \ldots$ etc in above equation

we get the value of $y(x_1)$ (or) y_1 .

Similarly expanding $y(x)$ in a Taylor series. about the point x_1

we will get $y_2 = y_1 + \frac{h}{1!} y'(x_1) + \frac{h^2}{2!} y''(x_1) + \frac{h^3}{3!} y'''(x_1) + \cdots$

Similarly expanding $y(x)$ at a general point x_n, we will get

$$y_{n+1} = y(x_{n+1}) = y_n + \frac{h}{1!} y_n' + \frac{h^2}{2!} y_n'' + \frac{h^3}{3!} y_n''' + \cdots$$

Working procedure :-

step(i) :- Compare the given diff. eqn. with $\frac{dy}{dx} = f(x,y)$, $y(x_0) = y_0$.

Identify $f(x,y)$, x_0 and y_0.

Identify the value of h (if not given)

step (ii) :-

Wkt Taylor series formula.

$$y_{n+1} = y(x_{n+1}) = y_n + \frac{h}{1!} y_n' + \frac{h^2}{2!} y_n'' + \cdots$$

Find y'', y''', y^{iv}, ...

step(iii) :- Find the values of y'', y''', y^{iv} at the point (x_0, y_0)

step (iv) :- sub. all these values in $y_1 = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \cdots$

We get the value of y_1 at $x = x_1$.

Similarly we find y_2 at $x = x_2$, y_3 at $x = x_3$ ...

Find $y(0.1)$ using Taylor's series method given that $\frac{dy}{dx} = 1+xy$ and $y(0) = 1$.

Sol:- Given that $\frac{dy}{dx} = 1+xy$ and $y(0) = 1$ —— ①

compare equation ① with $\frac{dy}{dx} = f(x,y)$, $y(x_0) = y_0$

Here $f(x,y) = 1+xy$, $x_0 = 0$, $y_0 = 1$.

We find the value of y at $x = 0.1$.

The difference between $x = 0.1$ and $x_0 = 0$ is 0.1

so $h = 0.1$

We have. $x_1 = x_0 + h = 0.1$

We find the value of y at $x_1 = 0.1$ i.e y_1 (or) $y(0.1)$

WKT Taylor's series

$$y_{n+1} = y(x_{n+1}) = y_n + \frac{h}{1!} y'_n + \frac{h^2}{2!} y''_n + \frac{h^3}{3!} y'''_n + \cdots$$

$$y' = 1+xy$$

Diff. w.r.t 'x', we get

$$y'' = xy' + y.$$

$$y''' = xy'' + y' + y' = xy'' + 2y'$$

$$y^{iv} = xy''' + y'' + 2y'' = xy''' + 3y''$$

To find y_1 (or) $y(0.1)$:-

Put $n=0$ in ②, $y_1 = y(x_1) = y_0 + \frac{h}{1!} y'_0 + \frac{h^2}{2!} y''_0 + \frac{h^3}{3!} y'''_0 + \frac{h^4}{4!} y^{iv}_0$ —— ③

At the pt $(x_0, y_0) = (0, 1)$

$$y'_0 = 1 + x_0 y_0 = 1.$$

$$y''_0 = x_0 y'_0 + y_0 = 1$$

$$y'''_0 = x_0 y''_0 + 2y'_0 = 0(1) + 2(1) = 2$$

$$y^{iv}_0 = x_0 y'''_0 + 3y''_0 = 0(2) + 3(1) = 3.$$

Substitute all these values in ①, we get.

$$y_1 = y(0.1) = 1 + \frac{0.1}{1!}(1) + \frac{(0.1)^2}{2!}(1) + \frac{(0.1)^3}{3!}(2) + \frac{(0.1)^4}{4!}(3)$$

$$y_1 = y(0.1) = 1 + 0.1 + 0.005 + 0.000333 + 0.0000125$$

$$= 1.105$$

$$\therefore \quad y(0.1) = 1.105.$$

Picards Method of Successive Approximations :-

Consider the differential equation $\frac{dy}{dx} = f(x,y)$ with initial condition $y(x_0) = y_0$

Integrating the diff. eqn. from x_0 to x.

$$\int_{x_0}^{x} dy = \int_{x_0}^{x} f(x,y)\, dx$$

$$[y]_{x_0}^{x} = \int_{x_0}^{x} f(x,y)\, dx$$

$$y(x) - y(x_0) = \int_{x_0}^{x} f(x,y)dx$$

$$y(x) = y(x_0) + \int_{x_0}^{x} f(x,y)\, dx \quad \text{--- (1)}$$

Here y is expressed under the integral sign, hence this is called an integral equation. This type of equations can be solved by the method of successive approximations.

The first approximation of y is obtained by putting $y = y_0$ in RHS of (1)

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)\, dx$$

Similarly the second approximation of y is obtained by substituting $y = y_1$ in RHS of (1)

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)\, dx$$

$$y_3 = y_0 + \int_{x_0}^{x} f(x, y_2)\, dx$$

Proceeding in this way, we obtain $y_4, y_5, y_6, \ldots y_n$.

$$y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1})\, dx.$$

Thus this method gives a sequence of approximations $y_1, y_2, y_3 \ldots y_n$

The process of iteration is stopped when the values of y_{n-1} and y_n are approximately equal.

→ Given the differential equation $\frac{dy}{dx} = \frac{x^2}{y^2+1}$ with initial condition $y=0$ at $x=0$. Use Picard's method to obtain y at $x=0.25$, $x=0.5$ and $x=1$.

Sol: Given that $\frac{dy}{dx} = \frac{x^2}{y^2+1}$ with Initial Condition $y=0$ at $x=0$ —— (1)

Compare eqn (1) with $\frac{dy}{dx} = f(x,y)$, $y(x_0) = y_0$.

Here $f(x,y) = \frac{x^2}{y^2+1}$, $x_0 = 0$, $y_0 = 0$.

WKT Picard's formula. $y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1})\, dx$.

<u>1st Approximation</u> :-

$n=1$, $y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)\, dx$.

$$y_1 = 0 + \int_0^x \frac{x^2}{y_0^2+1}\, dx$$

$$= \int_0^x x^2\, dx = \left[\frac{x^3}{3}\right]_0^x$$

$$y_1 = \frac{x^3}{3}$$

At $x=0.25$ $\quad y_1 = \frac{(0.25)^3}{3} = 0.005208$

At $x=0.5$ $\quad y_1 = \frac{(0.5)^2}{3} = 0.04166$

At $x=1$, $\quad y_1 = \frac{1}{3} = 0.333$.

<u>2nd Approximation</u> :-

$n=2$, $y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)\, dx$

$$y_2 = 0 + \int_0^x \frac{x^2}{y_1^2+1}\, dx = \int_0^x \frac{x^2}{1+\left(\frac{x^3}{3}\right)^2}\, dx.$$

Put $\frac{x^3}{3} = t$

$\frac{3x^2}{3} dx = dt \Rightarrow x^2 dx = dt$.

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$$y_2 = \int_0^{x^3/3} \frac{dt}{1+t^2} = \left[\tan^{-1}(t)\right]_0^{x^3/3}$$

$$y_2 = \tan^{-1}\left(\frac{x^3}{3}\right) - \tan^{-1}(0)$$

$$y_2 = \tan^{-1}\left(\frac{x^3}{3}\right)$$

At $x = 0.25$, $y_2 = \tan^{-1}\left(\frac{(0.25)^3}{3}\right) = \tan^{-1}(0.005208) = 0.005208$

At $x = 0.5$, $y_2 = \tan^{-1}\left(\frac{(0.5)^3}{3}\right) = \tan^{-1}(0.041667) = 0.04164$.

At $x = 1$, $y_2 = \tan^{-1}\left(\frac{1}{3}\right) = \tan^{-1}(0.333) = 0.32145$

1st. and 2nd approximations are approximately equal.

$\therefore$ $y(0.25) = 0.005208$

$y(0.5) = 0.04164$

$y(1) = 0.32145$

Euler's Method :-

Consider the differential equation $\frac{dy}{dx} = f(x,y)$ with initial condition $y(x_0) = y_0$.

$$dy = f(x,y)\, dx \quad \text{--- } (1)$$

Let us suppose that we want to find the approximate value of y say y_n at $x = x_n$ we divide the interval $[x_0, x_n]$ into n subintervals of equal length h say $x_0, x_1, x_2, \ldots x_n$.

Integrating (1) over $[x_0, x_1]$

$$\int_{x_0}^{x_1} dy = \int_{x_0}^{x_1} f(x,y)\, dx.$$

$$\left[y(x)\right]_{x_0}^{x_1} = \int_{x_0}^{x_1} f(x,y)\, dx$$

$$y(x_1) - y(x_0) = \int_{x_0}^{x_1} f(x,y)\, dx.$$

$$y_1 = y_0 + \int_{x_0}^{x_1} f(x,y)\, dx.$$

Let us put $f(x,y) = f(x_0, y_0)$ then

$$y_1 = y_0 + \int_{x_0}^{x_1} f(x_0, y_0)\, dx.$$

$$y_1 = y_0 + f(x_0, y_0)\left[x\right]_{x_0}^{x_1}$$

$$y_1 = y_0 + f(x_0, y_0)(x_1 - x_0)$$

$$y_1 = y_0 + h f(x_0, y_0)$$

Similarly $y_2 = y_1 + h f(x_1, y_1)$

$$y_3 = y_2 + h f(x_2, y_2)$$

$$\ldots$$

$$y_n = y_{n-1} + h f(x_{n-1}, y_{n-1}).$$

Working Procedure :-

Step (i) :- Compare given diff. eqn. with $\frac{dy}{dx} = f(x,y)$ with initial condition $y(x_0) = y_0$.

Identify $f(x,y)$, x_0 and y_0.

Identify the value of h (if not given)

Step (ii) WKT Eulers formula $y_n = y_{n-1} + h f(x_{n-1}, y_{n-1})$

$n=1$, $y_1 = y_0 + h f(x_0, y_0)$

Sub. the values of x_0, y_0 and h, we get y_1 (or) $y(x_1)$

$n=2$, $y_2 = y_1 + h f(x_1, y_1)$

Sub. the values of x_1, y_1 and h, we get y_2 (or) $y(x_2)$

Similarly we proceed in this we get $y_3, y_4 \dots$

Solve by Euler's method, the equation $\frac{dy}{dx} = x+y$, $y(0) = 0$ choose $h = 0.2$, compute $y(0.4)$ and $y(0.6)$

Sol: Given that $\frac{dy}{dx} = x+y$, $y(0) = 0$ —— ①

Compare eqn ① with $\frac{dy}{dx} = f(x,y)$, $y(x_0) = y_0$

Here $f(x,y) = x+y$

$x_0 = 0$

$y_0 = 0$

Given that $h = 0.2$

We have $x_1 = x_0 + h = 0 + 0.2 = 0.2$

$x_2 = x_1 + h = 0.2 + 0.2 = 0.4$

$x_3 = x_2 + h = 0.4 + 0.2 = 0.6$

→ We find the value of y at $x_1 = 0.2$ i.e y_1 or $y(0.2)$

→ We find the value of y at $x_2 = 0.4$ i.e y_2 or $y(0.4)$

→ We find the value of y at $x_3 = 0.6$ i.e y_3 or $y(0.6)$

We know that Euler's formula.

$$y_n = y_{n-1} + h f(x_{n-1}, y_{n-1})$$

To find y_1 (or) $y(0.2)$:-

$n=1$, $y_1 = y_0 + h f(x_0, y_0)$

$y_1 = 0 + (0.2)(x_0 + y_0)$

$y_1 = (0.2)(0+0) = 0$

$\therefore y_1 = y(0.2) = 0.$

To find y_2 (or) $y(0.4)$:-

$n=2$, $y_2 = y_1 + h f(x_1, y_1)$

$y_2 = 0 + (0.2)(x_1 + y_1)$

$$y_2 = (0.2)(0.2+0)$$

$$y_2 = 0.04$$

$$\therefore \quad y_2 = y(0.4) = 0.04.$$

<u>To find</u> <u>y_3</u> (or) <u>$y(0.6)$</u> :-

$n=3,$ $$y_3 = y_2 + h \cdot f(x_2, y_2)$$

$$y_3 = (0.04) + (0.2)(x_2 + y_2)$$

$$y_3 = (0.04) + (0.2)(0.4 + 0.04)$$

$$y_3 = 0.128$$

$$\therefore \quad y_3 = y(0.6) = 0.128.$$

Modified Euler's Method :-

Consider the differential eqn. $\frac{dy}{dx} = f(x, y)$ with Initial condition $y(x_0) = y_0$.

For find y_n at $x = x_n$ the modified Euler's formula is given by

$$y_r^{(n)} = y_{r-1} + \frac{h}{2}\left[f(x_{r-1}, y_{r-1}) + f(x_r, y_r^{(n-1)})\right]$$

To find $y(x_1) = y_1$ at $x = x_1 = x_0 + h$.

$r = 1, \quad n = 1, 2, 3 \ldots$

Using Euler's formula, $y_1^{(0)} = y_0 + h f(x_0, y_0)$

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(2)})\right]$$

$$\ldots \ldots \ldots$$

$$y_1^{(n+1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(n)})\right]$$

If two successive values of $y_1^{(k)}, y_1^{(k+1)}$ are sufficiently close to one another we will take the common values as y_1.

Now we have $\frac{dy}{dx} = f(x, y)$ with $y = y_1$ at $x = x_1$.

To get $y_2 = y(x_2) = y(x_1 + h)$ we use the above procedure again.

(1) Solve by Euler's modified method the equation $\frac{dy}{dx} = x+y$, $y(0)=0$ choose $h=0.2$ compute $y(0.4)$

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Sol:- Given the differential equation is $\frac{dy}{dx} = x+y$

The initial condition is $y(0)=0$

$x_0 = 0, \; y_0 = 0$

Modified Euler's formula is

$$y_r^{(n)} = y_{r-1} + \frac{h}{2}\left[f(x_{r-1}, y_{r-1}) + f(x_r, y_r^{(n-1)})\right].$$

To find y_1 :-

$$x_1 = x_0 + h = 0+0.2 = 0.2$$

By Euler's formula, $y_1^{(0)} = y_0 + h f(x_0, y_0)$

$$y_1^{(0)} = 0 + (0.2)(x_0+y_0) = (0.2)(0+0) = 0.$$

$$y_1^{(0)} = 0.$$

$y_1^{(1)}$ First Approximation :-

$$n=1, \quad y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0,y_0) + f(x_1, y_1^{(0)})\right]$$

$$= y_0 + \frac{h}{2}\left[x_0 + y_0 + x_1 + y_1^{(0)}\right]$$

$$= 0 + \frac{0.2}{2}\left[0+0+0.2+0\right] = 0.02$$

$$y_1^{(1)} = 0.02$$

Second Approximation :-

$$n=2, \quad y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0,y_0) + f(x_1, y_1^{(1)})\right]$$

$$= y_0 + \frac{h}{2}\left[x_0 + y_0 + x_1 + y_1^{(1)}\right]$$

$$= 0 + \frac{0.2}{2}\left[0+0+0.2+0.02\right] = 0.1\left[0.22\right] = 0.022$$

$$y_1^{(2)} = 0.022$$

Third Approximation :-

$$n=3 \quad y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0,y_0) + f(x_1, y_1^{(2)})\right]$$

$$= 0 + \frac{0.2}{2}\left[x_0+y_0+x_1+y_1^{(2)}\right] = 0.1\left[0+0+0.2+0.022\right]$$

$$y_1^{(3)} = 0.1(0.222) = 0.0222.$$

At $x = 0.2$, second and third Approximations are approximately equal.

$\therefore y(0.2) = 0.0222$

To find y_2 :-

$x_2 = x_1 + h = 0.2 + 0.2 = 0.4$

By Euler's formula. $y_2^{(0)} = y_1 + h f(x_1, y_1)$

$y_2^{(0)} = 0.0222 + (0.2)(0.2 + 0.0222)$

$y_2^{(0)} = 0.06664$.

First Approximation :-

$r=2, \quad n=1, \qquad y_2^{(1)} = y_1 + \frac{h}{2}\left[f(x_1,y_1) + f(x_2, y_2^{(0)})\right]$

$\cancel{= 0.0222 + 0.2}$

$= y_1 + \frac{h}{2}\left[x_1 + y_1 + x_2 + y_2^{(0)}\right]$

$= 0.0222 + \frac{0.2}{2}\left[0.2 + 0.0222 + 0.4 + 0.06664\right]$

$= 0.0222 + 0.1\left[0.68884\right]$

$y_2^{(1)} = 0.091084$.

Second Approximation :-

$n=2 \qquad y_2^{(2)} = y_1 + \frac{h}{2}\left[f(x_1,y_1) + f(x_2, y_2^{(1)})\right]$

$= y_1 + \frac{h}{2}\left[x_1 + y_1 + x_2 + y_2^{(1)}\right]$.

$= 0.0222 + \frac{0.2}{2}\left[0.2 + 0.0222 + 0.4 + 0.091084\right]$

$= 0.0222 + 0.1\left[0.713284\right.$

$y_2^{(2)} = 0.0935284$.

Third Approximation :-

$n=3 \quad y_2^{(3)} = y_1 + \frac{h}{2}\left[f(x_1,y_1) + f(x_2, y_2^{(2)})\right]$

$= y_1 + \frac{h}{2}\left[x_1 + y_1 + x_2 + y_2^{(2)}\right]$

$= 0.0222 + \frac{0.2}{2}\left[0.2 + 0.0222 + 0.4 + 0.0935284\right]$

$= 0.0222 + 0.1\left[0.7157284\right] = 0.09377284$

At $x = 0.4$, second and third approximations are approximately equal. $\therefore y(0.4) = 0.0938$

(8) Given $\frac{dy}{dx} - \sqrt{xy} = 2$ and $y(1) = 1$, Find the value of $y(1.5)$ in steps of 0.25 using Euler's modified method. 183 (15)

Sol: G.T the differential eqn is $\frac{dy}{dx} - \sqrt{xy} = 2$

$$\frac{dy}{dx} = 2 + \sqrt{xy}$$

The Initial condition is $y(1) = 1$

$x_0 = 1 \quad y_0 = 1.$

Here $f(x,y) = 2 + \sqrt{xy} \quad h = 0.25$.

Modified Euler's formula

$$y_r^{(n)} = y_{r-1} + \frac{h}{2}\left[f(x_{r-1}, y_{r-1}) + f(x_r, y_r^{(n-1)})\right].$$

<u>To find y_1</u> :-

$$x_1 = x_0 + h = 1 + 0.25 = 1.25.$$

By Euler's formula, $y_1^{(0)} = y_0 + hf(x_0, y_0)$

$$y^{(0)} = y_0 + h(2 + \sqrt{x_0 y_0})$$

$$y_1^{(0)} = 1 + 0.25(2 + \sqrt{1}) = 1 + 0.25(3) = 1.75$$

<u>First Approximation</u> :-

$n = 1 \quad y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$

$$= y_0 + \frac{h}{2}\left[2 + \sqrt{x_0 y_0} + 2 + \sqrt{x_1 y_1^{(0)}}\right]$$

$$= 1 + \frac{0.25}{2}\left[2 + \sqrt{(1)(1)} + 2 + \sqrt{(1.75)(1.25)}\right]$$

$$= 1 + \frac{0.25}{2}\left[5 + \sqrt{(1.75)(1.25)}\right]$$

$$y_1^{(1)} = 1.8099.$$

<u>Second Approximation</u> :-

$n = 2, \quad y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[2 + \sqrt{x_0 y_0} + 2 + \sqrt{x_1 y_1^{(1)}}\right]$$

$$= 1 + \frac{0.25}{2}\left[4 + \sqrt{(1)(1)} + \sqrt{(1.25)(1.8099)}\right]$$

$$= 1 + \frac{0.25}{2}\left[5 + \sqrt{(1.25)(1.8099)}\right]$$

$$y_1^{(2)} = 1.8130.$$

Third Approximation :-

$n=3 \quad y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0,y_0) + f(x_1, y_1^{(2)})\right]$

$= y_0 + \frac{h}{2}\left[2+\sqrt{x_0 y_0} + 2+\sqrt{x_1 y_1^{(2)}}\right]$

$= 1 + \frac{0.25}{2}\left[4+\sqrt{(1)(1)} + \sqrt{(1.25)(1.8130)}\right]$

$= 1 + \frac{0.25}{2}\left[5 + \sqrt{(1.25)(1.8130)}\right]$

$y_1^{(3)} = 1.8132$

At $x = 1.25$, second and third approximations are approximately equal.

$\therefore \; y(1.25) = 1.8132$.

To find y_2 :-

$x_2 = x_1 + h = 1.25 + 0.25 = 1.5$.

$y_2^{(0)} = y_1 + h f(x_1, y_1)$

$= y_1 + h\left(2+\sqrt{x_1 y_1}\right)$

$= 1.8132 + (0.25)\left(2+\sqrt{(1.25)(1.8132)}\right)$

$= 1.8132 + (0.25)\left(2+\sqrt{(1.25)(1.8132)}\right)$

$y_2^{(0)} = 2.6896$.

First Approximation :—

$n=1, \quad y_2^{(1)} = y_1 + \frac{h}{2}\left[f(x_1,y_1) + f(x_2, y_2^{(0)})\right]$

$= y_1 + \frac{h}{2}\left[2+\sqrt{x_1 y_1} + 2+\sqrt{x_2 y_2^{(0)}}\right]$

$= 1.8132 + \frac{0.25}{2}\left[4+\sqrt{(1.25)(1.8132)} + \sqrt{(1.5)(2.6896)}\right]$

$y_2^{(1)} = 2.7525$

Second Approximation :—

$n=2, \quad y_2^{(2)} = y_1 + \frac{h}{2}\left[f(x_1,y_1) + f(x_2, y_2^{(1)})\right]$

$= y_1 + \frac{h}{2}\left[2+\sqrt{x_1 y_1} + 2+\sqrt{x_2 y_2^{(1)}}\right]$

$= 1.8132 + \frac{0.25}{2}\left[4+\sqrt{(1.25)(1.8132)} + \sqrt{(1.5)(2.7525)}\right]$

$y_2^{(2)} = 2.7554$

Third Approximation :-

$n=3 \quad y_2^{(3)} = y_1 + \frac{h}{2}\left[f(x_1,y_1) + f(x_2,y_2^{(2)})\right]$

$= y_1 + \frac{h}{2}\left[2+\sqrt{x_1 y_1} + 2 + \sqrt{x_2 y_2^{(2)}}\right]$

$= 1.8132 + \frac{0.25}{2}\left[2+\sqrt{(1.25)(1.8132)} + 2 + \sqrt{(1.5)(2.7554)}\right]$

$= 1.8132 + \frac{0.25}{2}\left[4+\sqrt{(1.25)(1.8132)} + \sqrt{(1.5)(2.7554)}\right]$

$y_2^{(3)} = 2.7555$

At $x=1.5$; second and third approximations are approximately equal.

$\therefore y(1.5) = 2.7555$

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Runge Kutta Methods :-

First order Runge kutta Method :-

We know that, By Euler's method.

$$y_1 = y_0 + h f(x_0, y_0)$$

$$= y_0 + h y_0' \quad (\because y' = f(x,y))$$

By Taylor's series

$$y_1 = y_0 + h y_0' + \frac{h^2}{2!} y_0'' + \dots$$

Euler's method is same as the Taylor's series solution upto the term in h.

$\Rightarrow$ Euler's method is Runge Kutta method of first order.

Second order Runge Kutta Method :-

Consider the differential equation $\frac{dy}{dx} = f(x,y)$ with Initial condition $y(x_0) = y_0$.

$$\frac{dy}{dx} = f(x,y)$$

$$dy = f(x,y)\,dx.$$

Integrating the above equation over $[x_0, x_1]$ using Trapezoidal rule.

$$\int_{x_0}^{x_1} dy = \int_{x_0}^{x_1} f(x,y)\,dx$$

$$[y]_{x_0}^{x_1} = \frac{h}{2}\left[f(x_0,y_0) + f(x_1,y_1)\right]$$

$$y(x_1) - y(x_0) = \frac{h}{2}\left[f(x_0,y_0) + f(x_1,y_1)\right].$$

$$y_1 = y_0 + \frac{h}{2}\left[f(x_0,y_0) + f(x_0+h,\ y_0 + h f(x_0,y_0))\right]$$

$$y_1 = y_0 + \frac{1}{2}\left[h f(x_0,y_0) + h f(x_0+h,\ y_0 + h f(x_0,y_0))\right] \quad \text{--- (1)}$$

$x_1 = x_0 + h \qquad y_1 = y_0 + h f(x_0,y_0).$

Now we put $k_1 = h f(x_0,y_0)$.

$$k_2 = h f(x_0+h, y_0 + h f(x_0,y_0))$$

$$k_2 = h f(x_0+h, y_0 + k_1).$$

Then from (1).

$$y_1 = y_0 + \frac{1}{2}(k_1 + k_2)$$

similarly for finding y_2, $y_2 = y_1 + \frac{1}{2}(k_1 + k_2)$

where $k_1 = h f(x_1, y_1)$

$k_2 = h f(x_1 + h, y_1 + k_1)$

In General. $y_{n+1} = y_n + \frac{1}{2}(k_1 + k_2)$

where $k_1 = h f(x_n, y_n)$

$k_2 = h f(x_n + h, y_n + k_1)$ $n = 0, 1, 2, \ldots$

which is called second order Runge Kutta formula.

(II) Fourth order

Third order Runge Kutta Method:—

The third order Runge Kutta method formula is.

$$y_{n+1} = y_n + \frac{(k_1 + 4k_2 + k_3)}{6} \quad n = 0, 1, 2, \ldots$$

where $k_1 = h f(x_n, y_n)$

$k_2 = h f(x_n + \frac{h}{2}, y_n + \frac{k_1}{2})$.

$k_3 = h f(x_n + h, y_n + 2k_2 - k_1)$.

Fourth order Runge Kutta Method:—

The fourth order Runge Kutta method formula is

$$y_{n+1} = y_n + \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]$$

$n = 0, 1, 2, \ldots$

where $k_1 = h f(x_n, y_n)$.

$k_2 = h f(x_n + \frac{h}{2}, y_n + \frac{k_1}{2})$

$k_3 = h f(x_n + \frac{h}{2}, y_n + \frac{k_2}{2})$

$k_4 = h f(x_n + h, y_n + k_3)$.

(1) Given $\frac{dy}{dx} = -y$, $y(0)=1$, using Runge Kutta method of second order find the value of y at $x=0.1$ at $x=0.2$.

Sol: Given the differential equation is $\frac{dy}{dx} = -y$.

Here $f(x,y) = -y$

The initial condition is $y(0)=1$

$x_0 = 0 \quad y_0 = 1$.

The second order Runge Kutta formula is.

$$y_{n+1} = y_n + \frac{1}{2}(k_1+k_2)$$

$$k_1 = h\, f(x_n, y_n)$$

$$k_2 = h\, f(x_n+h, y_n+k_1)$$

<u>To find y_1 :-</u>

$n=0$, $\quad y_1 = y_0 + \frac{1}{2}(k_1+k_2)$

$$k_1 = h\, f(x_0, y_0)$$

$$k_2 = hf(x_0+h, y_0+k_1)$$

Here $h = 0.1$,

$$k_1 = h(-y_0)$$

$$k_1 = (0.1)(-1) = -0.1$$

$$k_2 = h(-(y_0+k_1)) = h(-y_0-k_1)$$

$$k_2 = 0.1(-1+0.1) = (0.1)(-0.9)$$

$$k_2 = -0.09.$$

$$y_1 = y(0.1) = y_0 + \frac{1}{2}(k_1+k_2)$$

$$= 1 + \frac{1}{2}(-0.1-0.09)$$

$$= 1 + \frac{1}{2}(-0.19)$$

$$y_1 = 0.905$$

<u>To find y_2 :-</u>

$n=1$, $\quad y_2 = y_1 + \frac{1}{2}(k_1+k_2)$

$$k_1 = h\, f(x_1, y_1).$$

$$k_2 = h\, f(x_1+h, y_1+k_1).$$

$$k_1 = h(-y_1)$$

$$k_1 = (0.1)(-0.905) = -0.0905$$

$K_2 = h \cdot f(x_1+h, y_1+K_1)$

$= h(-y_1 - K_1)$

$= (0.1)(-0.905 + 0.0905)$

$K_2 = -0.08145$

$y_2 = y(0.2) = y_1 + \frac{1}{2}(K_1+K_2)$

$y_2 = 0.905 + \frac{1}{2}(-0.0905 - 0.08145)$

$y_2 = 0.819025$

(2) Using Runge Kutta method of second order, compute $y(2.5)$ from $\frac{dy}{dx} = \frac{x+y}{x}$ $y(2) = 2$ taking $h = 0.25$.

Sol:- G.T the differential equation is $\frac{dy}{dx} = \frac{x+y}{x}$

Here $f(x,y) = \frac{x+y}{x}$

The initial condition is $y(2) = 2$ i.e. $x_0 = 2$, $y_0 = 2$.

$h = 0.25$

The second order Euler's formula is.

$y_{n+1} = y_n + \frac{1}{2}(K_1+K_2)$

$K_1 = h f(x_n, y_n)$

$K_2 = h f(x_n+h, y_n+K_1)$

To find y_1 :-

$n=0$, $y_1 = y_0 + \frac{1}{2}(K_1+K_2)$

$K_1 = h f(x_0, y_0)$

$K_2 = h f(x_0+h, y_0+K_1)$

$K_1 = h f(x_0, y_0)$

$K_1 = h \cdot \left(\frac{x_0+y_0}{x_0}\right)$

$K_1 = (0.25)\left(\frac{2+2}{2}\right) = (0.25)2 = 0.5$

$K_2 = h f(x_0+h, y_0+K_1)$

$K_2 = h f(2.25, 2.5) = (0.25)\left(\frac{2.25+2.5}{2.25}\right)$

$K_2 = 0.528$

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$$y_1 = y(2.25) = y_0 + \frac{1}{2}(k_1+k_2)$$

$$y_1 = 2 + \frac{1}{2}(0.5+0.528)$$

$$y_1 = 2.514 .$$

To find y_2 :-

$n=2$, $y_2 = y_n + \frac{1}{2}(k_1+k_2)$

$$k_1 = h f(x_1, y_1)$$

$$k_2 = h f(x_1+h, y_1+k_1)$$

$$k_1 = h f(x_1, y_1)$$

$$k_1 = h \cdot \left(\frac{x_1+y_1}{x_1}\right)$$

$$= (0.25)\left(\frac{2.25+2.514}{2.25}\right)$$

$$k_1 = 0.5293$$

$$k_2 = h f(x_1+h, y_1+k_1)$$

$$k_2 = h f(2.25+0.25, 2.514+0.5293)$$

$$k_2 = h f(2.5, 3.0433)$$

$$k_2 = (0.25)\left(\frac{2.5+3.0433}{2.5}\right)$$

$$k_2 = 0.55433.$$

$$y_2 = \frac{1}{2}(k_1+k_2)+y_1$$

$$y_2 = \frac{1}{2}(0.5293+0.55433) + 2.514$$

$$y_2 = 3.055815 .$$

Obtain the values of y at $x=0.1$ and 0.2 using third order Range Kutta method, given $\frac{dy}{dx} = -y$, $y(0)=1$.

Sol: - G.T the differential eqn is $\frac{dy}{dx} = -y$

Here $f(x,y) = -y$.

The initial condition is $y(0)=1$.

Third order Runge kutta formula is

$$y_{n+1} = y_n + \frac{(k_1+4k_2+k_3)}{6} \quad n=0,1,2,\dots$$

where $k_1 = hf(x_n, y_n)$

$$k_2 = h f\left(x_n+\frac{h}{2}, y_n+\frac{k_1}{2}\right)$$

$$k_3 = h f(x_n+h, y_n+2k_2-k_1).$$

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To find y_1:-

$n=0, \quad y_1 = y_0 + \left(\frac{k_1+4k_2+k_3}{6}\right)$

$k_1 = hf(x_0, y_0)$

$k_2 = h f(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2})$

$k_3 = h f(x_0+h, y_0 + 2k_2 - k_1)$

$k_1 = hf(x_0, y_0)$

$k_1 = h(-y_0) = (0.1)(-1) = -0.1$

$k_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2})$

$k_2 = h(-(y_0 + \frac{k_1}{2})) = h(-y_0 - \frac{k_1}{2})$

$k_2 = (0.1)(-1 + \frac{0.1}{2})$

$k_2 = -0.095$.

$k_3 = hf(x_0+h, y_0+2k_2-k_1)$

$k_3 = h(-(y_0+2k_2-k_1))$

$k_3 = (0.1)(-(1-2(0.095)+0.1))$

$k_3 = (0.1)(-0.91)$

$k_3 = -0.091$.

$y_1 = y_0 + \left(\frac{k_1+4k_2+k_3}{6}\right)$

$= 1 + \frac{(-0.1+4(-0.095)-0.091)}{6}$

$y_1 = 0.905$

To find y_2:-

$x_1 = x_0 + h = 0 + 0.1 = 0.1 \quad y_1 = 0.905$

$n=1, \quad y_2 = y_1 + \left(\frac{k_1+4k_2+k_3}{6}\right)$

Where $k_1 = hf(x_1, y_1)$

$k_2 = hf(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2})$

$k_3 = hf(x_1+h, y_1+2k_2-k_1)$

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$K_1 = hf(x_1, y_1)$

$K_1 = h(-y_1) = (0.1)(-0.905)$

$K_1 = -0.0905$

$K_2 = hf(x_1 + \frac{h}{2}, y_1 + \frac{K_1}{2})$

$= h(-(y_1 + \frac{K_1}{2}))$

$= (0.1)(-(0.905 + \frac{(-0.0905)}{2}))$

$K_2 = -0.085975$

$K_3 = hf(x_n + h, y_n + 2K_2 - K_1)$

$K_3 = hf(x_1 + h, y_1 + 2K_2 - K_1)$

$K_3 = h(-(y_1 + 2K_2 - K_1))$

$K_3 = (0.1)(-(0.905 + 2(-0.085975) + 0.0905))$

$K_3 = -0.082355$

$y_2 = y(0.2) = y_1 + \frac{1}{6}(K_1 + 4K_2 + K_3)$

$= 0.905 + \frac{1}{6}(-0.0905 + 4(-0.085975) + 0.082355)$

$= 0.905 + \frac{1}{6}($

$y_2 = 0.818874.$

14) Given that $y' = y - x$, $y(0) = 2$ find $y(0.2)$ using Runge Kutta method Take $h = 0.1$.

Sol:- G.T the differential equation $y' = y - x$.

Here $f(x, y) = y - x$.

The initial condition is $y(0) = 2$

$x_0 = 0$, $y_0 = 2$, $h = 0.1$

Fourth order Runge Kutta formula is

$$y_{n+1} = y_n + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

$K_1 = hf(x_n, y_n)$

$K_2 = hf(x_n + \frac{h}{2}, y_n + \frac{K_1}{2})$

$K_3 = hf(x_n + \frac{h}{2}, y_n + \frac{K_2}{2})$

$K_4 = hf(x_n + h, y_n + K_3)$

To find y_1 :-

$n=0,\quad y_1 = y_0 + \frac{1}{6}(k_1+2k_2+2k_3+k_4)$

$k_1 = hf(x_0,y_0)$

$k_2 = hf(x_0+\frac{h}{2}, y_0+\frac{k_1}{2})$

$k_3 = hf(x_0+\frac{h}{2}, y_0+\frac{k_2}{2})$

$k_4 = hf(x_0+h, y_0+k_3)$

$\rightarrow k_1 = hf(x_0,y_0)$

$k_1 = h(y_0-x_0)$

$k_1 = (0.1)(2-0) = 0.2$

$\rightarrow k_2 = hf(x_0+\frac{h}{2}, y_0+\frac{k_1}{2})$

$= hf(0+\frac{0.1}{2}, 2+\frac{0.2}{2})$

$= hf(0.05, 2.1)$

$= (0.1)(2.1-0.05)$

$k_2 = 0.205$

$\rightarrow k_3 = hf(x_0+\frac{h}{2}; y_0+\frac{k_2}{2})$

$= hf(0+\frac{0.1}{2}, 2+\frac{0.205}{2})$

$k_3 = hf(0.05, 2.10025)$

$k_3 = (0.1)(2.0525-0.05)$

$k_3 = 0.20525$

$\rightarrow k_4 = hf(x_0+h, y_0+k_3)$

$k_4 = hf(0.1, 2+0.20525)$

$k_4 = 0.210525$

$y_1 = y_0 + \frac{1}{6}(k_1+2(k_2+k_3)+k_4)$

$= 2+\frac{1}{6}(0.2+(0.205+0.20525)2+0.210525)$

$= 2+\frac{1}{6}($

$y_1 = 2.2052$

To find y_2 :-

$n=1, \quad y_2 = y_1 + \frac{(k_1 + 2k_2 + 2k_3 + k_4)}{6}$

$k_1 = hf(x_1, y_1)$

$k_2 = hf(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2})$

$k_3 = hf(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2})$

$k_4 = hf(x_1 + h, y_1 + k_3)$

$h = 0.1 \quad x_1 = 0.1 \quad y_1 = 2.2052$

$k_1 = hf(x_1, y_1)$

$k_1 = h(y_1 - x_1) = (0.1)(2.2052 - 0.1) = (0.1)(2.1052) = 0.21052$

$k_2 = hf(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2})$

$= hf(0.1 + \frac{0.1}{2}, 2.2052 + \frac{0.21052}{2})$

$= hf(0.15, 2.2052 + 0.10526) = hf(0.15, 2.31046)$

$= (0.1)(2.31046 - 0.15) = (0.1)(2.16046)$

$k_2 = 0.216046$

$k_3 = hf(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2})$

$= hf(0.1 + \frac{0.1}{2}, 2.2052 + \frac{0.216046}{2})$

$= hf(0.1 + 0.05, 2.2052 + 0.108023)$

$= hf(0.15, 2.313223)$

$k_3 = (0.1)(2.313223 - 0.15) = (0.1)(2.163223)$

$k_3 = 0.2163223$

$k_4 = hf(x_1 + h, y_1 + k_3)$

$= hf(0.1 + 0.1, 2.2052 + 0.2163223)$

$= hf(0.2, 2.4215223)$

$= (0.1)(2.4215223 - 0.2)$

$k_4 = 0.22215223$

$y_2 = y_1 + \left(\frac{k_1 + 2k_2 + 2k_3 + k_4}{6}\right)$

$y_2 = 2.2052 + \frac{(0.21052 + 2(0.216046 + 0.2163223) + 0.22215223)}{6}$

$y_2 = 2.421496483$

$y_2 = 2.4215$

Given $\frac{dy}{dx} = \frac{y-x}{y+x}$ $y(0)=1$, compute $y(0.2)$ by using fourth order Runge Kutta method by taking $h=0.2$.

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Sol: G/T $\frac{dy}{dx} = \frac{y-x}{y+x}$

Here $y(0)=1$

$x_0=0,\ y_0=1 \quad h=0.2$

Here $f(x,y) = \frac{y-x}{y+x}$

The fourth order Runge Kutta formula is

$$y_{n+1} = y_n + \frac{(k_1+2k_2+2k_3+k_4)}{6}$$

$$k_1 = hf(x_n, y_n)$$
$$k_2 = hf(x_n+\tfrac{h}{2}, y_n+\tfrac{k_1}{2})$$
$$k_3 = hf(x_n+\tfrac{h}{2}, y_n+\tfrac{k_2}{2})$$
$$k_4 = hf(x_n+h, y_n+k_3)$$

To find y_1 :-

$n=0$, $$y_1 = y_0 + \frac{(k_1+2k_2+2k_3+k_4)}{6}$$

$$k_1 = hf(x_0, y_0)$$
$$k_2 = hf(x_0+\tfrac{h}{2}, y_0+\tfrac{k_1}{2})$$
$$k_3 = hf(x_0+\tfrac{h}{2}, y_0+\tfrac{k_2}{2})$$
$$k_4 = hf(x_0+h, y_0+k_3)$$

$$k_1 = hf(x_0, y_0)$$
$$k_1 = h\left(\frac{y_0-x_0}{y_0+x_0}\right)$$
$$k_1 = 0.2\left(\frac{1-0}{1+0}\right) = 0.2$$

$$k_2 = hf(x_0+\tfrac{h}{2}, y_0+\tfrac{k_1}{2})$$
$$k_2 = hf\left(0+\frac{0.2}{2}, 1+\frac{0.2}{2}\right) = hf(0.1, 1.1)$$
$$k_2 = (0.2)\left(\frac{1.1-0.1}{1.1+0.1}\right) = (0.2)\left(\frac{1}{1.2}\right) = 0.16667$$

$$k_3 = hf(x_0+\tfrac{h}{2}, y_0+\tfrac{k_2}{2})$$
$$= hf\left(0+\frac{0.2}{2}, 1+\frac{0.16667}{2}\right) = hf(0.1, 1.083335)$$
$$k_3 = (0.2)\left(\frac{1.083335-0.1}{1.083335+0.1}\right) = (0.2)\left(\frac{0.983335}{1.183335}\right) = 0.166197$$

$$K_4 = h f(x_0 + h,\ y_0 + K_3)$$

$$= hf(0 + 0.2,\ 1 + 0.166197)$$

$$= hf(0.2,\ 1.166197)$$

$$K_4 = (0.2)\left(\frac{1.166197 - 0.2}{1.166197 + 0.2}\right) = (0.2)\left(\frac{0.966197}{1.366197}\right)$$

$$K_4 = 0.14144.$$

$$y_1 = y_0 + \frac{(K_1 + 2K_2 + 2K_3 + K_4)}{6}$$

$$= 1 + \frac{0.2 + 2(0.16667 + 0.166197) + 0.14144)}{6}$$

$$= 1 + \cancel{0.5035887}\ 0.16786$$

$$y_1 = \cancel{1.5035887}\ 1.16786.$$

Numerical Integration :—

The method finding the value of an integral of the form $\int_a^b f(x)dx$ using numerical techniques is called Numerical integration.

Let $y = f(x)$ be a function on $[a,b]$. If the function $f(x)$ is defined explicitly and the integral of $f(x)$ can be calculated by the usual methods then the definite integral $\int_a^b f(x)\,dx$ can be found easily.

If the function is given in tabular form and the integral of $f(x)$ is difficult to find then numerical integration is needed.

Numerical integration is used to obtain approximate answers for definite integrals that can not be solved analytically. It is a process of finding the numerical value of a definite integral $I = \int_a^b f(x)dx$ when the function $y = f(x)$ is not known explicitly.

Importance of Numerical Methods :—

The Numerical methods are important because finding an analytical procedure to solve an equation may not be always available. In such cases Numerical Analysis provides approximate solutions.

(i) To solve the ordinary differential equations.

(ii) To solve the Algebraic and Transcendental Equations.

(iii) To solve Numerical Integration and Differentiation.

Trapezoidal Rule :-

$$\int_{x_0}^{x_n} y\,dx = \frac{h}{2}\left(y_0 + y_1 + y_2 + y_3 + y_4 + \cdots + y_{n-1} + y_n\right)$$

$$\int_{x_0}^{x_n} y\,dx = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + y_3 + \cdots + y_{n-1})\right]$$

where $h = \frac{b-a}{n} = \frac{x_n - x_0}{n}$

(1) Calculate the value $\int_0^1 \frac{x}{1+x}\,dx$ correct to 3 significant figures taking 6 intervals by trapezoidal rule.

Sol:- Let $y = \frac{x}{1+x}$, $x_0 = 0$ $x_n = 1$ $n = 6$

$$h = \frac{b-a}{n} = \frac{x_n - x_0}{n} = \frac{1-0}{6} = \frac{1}{6}.$$

x	$y = \frac{x}{1+x}$
$x_0 = 0$	$y_0 = \frac{x_0}{1+x_0} = \frac{0}{1+0} = 0$
$x_1 = \frac{1}{6}$	$y_1 = \frac{x_1}{1+x_1} = \frac{1/6}{1+1/6} = \frac{1}{7} = 0.1428$
$x_2 = \frac{1}{3}$	$y_2 = \frac{x_2}{1+x_2} = \frac{1/3}{1+\frac{1}{3}} = \frac{1}{4} = 0.25$
$x_3 = \frac{1}{2}$	$y_3 = \frac{x_3}{1+x_3} = \frac{1/2}{1+1/2} = \frac{1}{3} = 0.333$
$x_4 = \frac{2}{3}$	$y_4 = \frac{x_4}{1+x_4} = \frac{2/3}{1+2/3} = \frac{2}{5} = 0.4$
$x_5 = \frac{5}{6}$	$y_5 = \frac{x_5}{1+x_5} = \frac{5/6}{1+5/6} = \frac{5}{11} = 0.4545$
$x_6 = 1$	$y_6 = \frac{x_6}{1+x_6} = \frac{1}{1+1} = \frac{1}{2} = 0.5$

Trapezoidal Rule.

$$\int_{x_0}^{x_n} y\,dx = \frac{h}{2}\left[(y_0+y_n)+2(y_1+y_2+y_3+\cdots+y_{n-1})\right]$$

$$\int_0^1 \frac{x}{1+x}\,dx = \frac{h}{2}\left[(y_0+y_6)+2(y_1+y_2+y_3+\cdots+y_5)\right]$$

$$= \frac{1}{12}\left[(0+0.5)+2(0.1428+0.25+0.333+0.4+0.4545)\right]$$

$$\int_0^1 \frac{x}{1+x}\,dx = 0.30505$$

Simpson's $\frac{1}{3}$ - Rule :—

$$\int_{x_0}^{x_n} y\,dx = \frac{h}{3}\left[(y_0+y_n)+4(y_1+y_3+y_5+\cdots+y_{n-1})+2(y_2+y_4+\cdots+y_{n-2})\right]$$

$$\int_{x_0}^{x_n} y\,dx = \frac{h}{3}\left[(y_0+y_n)+4(\text{sum of odd ordinates})+2(\text{sum of even ordinates})\right].$$

Note:- It can be applied only when the given interval $[a,b]$ is subdivided into even no. of sub intervals each of width h and with in any two consecutive subintervals the interpolating polynomial $\phi(x)$ is of degree 2.

(1) The velocity of a train which starts from rest is given by the following table.

t_{min}	2	4	6	8	10	12	14	16	18	20
$V_{km/hr}$	16	28.8	40	46.4	51.2	32	17.6	8	3.2	0

Estimate total distance run in 20 min.

Sol:- We know that
The rate of change of displacement is called velocity.

i.e $\frac{ds}{dt} = v$

$$ds = v\,dt$$

$$s = \int_0^{20} v\,dt$$

Since the train starts from rest $\Rightarrow$ $v=0$ at $t=0$

$t_0 = 0,\ v_0 = 0.$

$t_0 = 0 \qquad t_n = 20.$

$$h = \frac{20-0}{10} = 2 \text{ min} = \frac{2}{60} = \frac{1}{30} \text{ hrs.}$$

Simpson's $\frac{1}{3}$ Rule.

$$s = \int_0^{20} v\,dt = \frac{h}{3}\left[(y_0+y_{10}) + 4(y_1+y_3+y_5+y_7+y_9) + 2(y_2+y_4+y_6+y_8)\right]$$

$$= \frac{1}{90}\left[(0+0) + 4(16+40+51.2+17.6+3.2) + 2(28.8+46.4+32+8)\right]$$

$$= \frac{1}{90}\left[4(128) + 2(115.2)\right]$$

$$= 8.25 \text{ km}$$

The distance run by train in 20 min $= 8.25$ km..

Simpson's $\frac{3}{8}$ - Rule :—

$$\int_{x_0}^{x_n} y\,dx = \frac{3h}{8}\left[(y_0+y_n) + 3(y_1+y_2+y_4+\cdots+y_{n-1}) + 2(y_3+y_6+y_9+\cdots+y_{n-3})\right]$$

Note:- It can be applied when the range $[a,b]$ is divided into no. of sub intervals, which is a multiple of 3.

(1) Evaluate $\int_0^1 \frac{1}{1+x^2}\,dx$ using Simpson's $\frac{1}{3}$ and Simpson's $\frac{3}{8}$ Rule, and Trapezoidal Rule. Also compare it with its exact value. Hence obtain the approximate value of π in each.

sol: Let $y = f(x) = \frac{1}{1+x^2}$.

Given $x_0 = 0$ $x_n = 1$ $h = \frac{x_n - x_0}{n} = \frac{1-0}{6} = \frac{1}{6}$.

$n = 6$

x	$y = f(x) = \frac{1}{1+x^2}$
$x_0 = 0$	$y_0 = \frac{1}{1+x_0^2} = \frac{1}{1+0} = 1$
$x_1 = \frac{1}{6}$	$y_1 = \frac{1}{1+x_1^2} = \frac{36}{37} = 0.97297$
$x_2 = \frac{1}{3}$	$y_2 = \frac{1}{1+x_2^2} = \frac{9}{10} = 0.9$
$x_3 = \frac{1}{2}$	$y_3 = \frac{1}{1+x_3^2} = \frac{4}{5} = 0.8$
$x_4 = \frac{2}{3}$	$y_4 = \frac{1}{1+x_4^2} = \frac{9}{13} = 0.6923$
$x_5 = \frac{5}{6}$	$y_5 = \frac{1}{1+x_5^2} = \frac{36}{61} = 0.5902$
$x_6 = 1$	$y_6 = \frac{1}{1+x_6^2} = \frac{1}{1+1} = 0.5$

Trapezoidal Rule.

$$\int_{x_0}^{x_n} y\,dx = \frac{h}{2}\left[(y_0+y_n) + 2(y_1+y_2+y_3+\cdots+y_{n-1})\right]$$

$$\int_0^1 \frac{1}{1+x^2}\,dx = \frac{h}{2}\left[(y_0+y_6) + 2(y_1+y_2+y_3+y_4+y_5)\right]$$

$$= \frac{1}{12}\left[(1+0.5)+2(0.97297+0.9+0.8+0.6923+0.5902)\right]$$

$$= \frac{1}{12}\left[1.5 + 2(3.95547)\right]$$

$$= \frac{1}{12}\left[1.5 + 7.91094\right] = \frac{1}{12}(9.41094)$$

$$\int_0^1 \frac{1}{1+x^2}\,dx = 0.784245$$

Simpson's $\frac{1}{3}$ Rule :—

$$\int_{x_0}^{x_n} y\,dx = \frac{h}{3}\left[(y_0+y_n) + 4(y_1+y_3+y_5+\cdots) + 2(y_2+y_4+y_6+\cdots)\right]$$

$$\int_0^1 \frac{1}{1+x^2}\,dx = \frac{h}{3}\left[(y_0+y_6) + 4(y_1+y_3+y_5) + 2(y_2+y_4)\right]$$

$$= \frac{1}{18}\left[(1+0.5) + 4(0.97297+0.8+0.5902) + 2(0.9+0.6923)\right]$$

$$= \frac{1}{18}\left[1.5 + 4(2.36317) + 2(1.5923)\right]$$

$$= \frac{1}{18}\left[1.5 + 9.45268 + 3.1846\right] = \frac{1}{18}(14.13728)$$

$$\int_0^1 \frac{1}{1+x^2}\,dx = 0.785404444$$

Simpson's $\frac{3}{8}$ Rule :-

$$\int_{x_0}^{x_n} y\,dx = \frac{3h}{8}\left[(y_0+y_n) + 3(y_1+y_2+y_4+y_5+\cdots) + 2(y_3+y_6+\cdots)\right]$$

$$\int_0^1 \frac{1}{1+x^2}\,dx = \frac{3h}{8}\left[(y_0+y_6) + 3(y_1+y_2+y_4+y_5) + 2(y_3)\right].$$

$$= \frac{3}{48}\left[(1+0.5) + 3(0.97297+0.9+0.6923+0.5902) + 2(0.8)\right]$$

$$= \frac{1}{16}\left[1.5 + 9.46641 + 1.6\right] = \frac{1}{16}(12.56641)$$

$$\int_0^1 \frac{1}{1+x^2}\,dx = 0.785400625.$$

By direct integration

$$\int_0^1 \frac{1}{1+x^2}\,dx = \left[\tan^{-1}x\right]_0^1 = \tan^{-1}(1) - \tan^{-1}(0)$$

$$= \tan^{-1}\left(\tan\frac{\pi}{4}\right) - \tan^{-1}(\tan 0)$$

$$\int_0^1 \frac{1}{1+x^2}\,dx. = \frac{\pi}{4} = 0.78571.$$

Error :- ① (Trapezoidal Rule)

Error = Exact value - obtained value.

$= 0.78571 - 0.784245 = 0.001465.$

② Simpson's $\frac{1}{3}$ Rule.

Error = Exact value - obtained value.

$= 0.78571 - 0.7854044. = 0.0003056.$

③ Simpson's $\frac{3}{8}$ Rule.

Error = Exact value - obtained value.

$= 0.78571 - 0.785400625 = 0.000309375.$

π value due to simpson's $\frac{1}{3}$ Rule. Trapezoidal Rule.

$0.784245 = \frac{\pi}{4} \Rightarrow \pi = 3.13698$

π value due to simpson's $\frac{1}{3}$ Rule.

$0.7854044 = \frac{\pi}{4} \Rightarrow \pi = 3.1416176.$

π value due to simpson's $\frac{3}{8}$ Rule.

$$0.785400625 = \frac{\pi}{4} \Rightarrow \pi = 3.1416025$$

Evaluate $\int_0^{\pi} \sin x \, dx$ by dividing the range into 16 equal parts using (i) Trapezoidal rule (ii) Simpson's $\frac{1}{3}$rd rule (iii) Simpson's $\frac{3}{8}$ rule.

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Sol:- Let $y = f(x) = \sin x$.

Here $a = 0$ $b = \pi$

Number of sub intervals $n = 6$.

$$h = \frac{b-a}{n} = \frac{\pi - 0}{6} = \frac{\pi}{6}.$$

$x_0 = 0$, $x_1 = x_0 + h = \frac{\pi}{6}$ $x_2 = x_1 + h = \frac{\pi}{3}$ $x_3 = x_2 + h = \frac{\pi}{2}$.

$x_4 = x_3 + h = \frac{2\pi}{3}$ $x_5 = x_4 + h = \frac{5\pi}{6}$ $x_6 = x_5 + h = \pi$.

$y_0 = \sin x_0 = \sin 0 = 0$. $y_1 = \sin x_1 = \sin \frac{\pi}{6} = 0.5$.

$y_2 = \sin x_2 = \sin \frac{\pi}{3} = 0.866$ $y_3 = \sin x_3 = \sin \frac{\pi}{2} = 1$

$y_4 = \sin x_4 = \sin \frac{2\pi}{3} = 0.866$ $y_5 = \sin x_5 = \sin \frac{5\pi}{6} = 0.5$

$y_6 = \sin x_6 = \sin \pi = 0$.

(i) Trapezoidal rule :-

$$\int_0^{\pi} \sin x \, dx = \frac{h}{2}\left[(y_0 + y_6) + 2(y_1 + y_2 + y_3 + y_4 + y_5)\right].$$

$$= \frac{\pi}{12}\left[(0+0) + 2(0.5 + 0.866 + 1 + 0.866 + 0.5)\right]$$

$$= \frac{\pi}{12}[7.464] = \frac{22}{7 \times 12}[7.464] = 1.95486$$

$$\therefore \int_0^{\pi} \sin x \, dx = 1.95486.$$

(ii) Simpson's $\frac{1}{3}$rd rule :-

$$\int_0^{\pi} \sin x \, dx = \frac{h}{3}\left[(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)\right].$$

$$= \frac{\pi}{18}\left[(0+0) + 4(0.5+1+0.5) + 2(0.866+0.866)\right]$$

$$= \frac{\pi}{18}\left[8 + 3.464\right] = \frac{22}{7\times 18}(11.464) = 2.0017.$$

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$$\therefore \int_0^{\pi} \sin x \, dx = 2.0017.$$

(iii) Simpson's $\frac{3}{8}$ rule :-

$$\int_0^{\pi} \sin x \, dx = \frac{3h}{8}\left[(y_0+y_6) + 3(y_1+y_2+y_4+y_5) + 2(y_3)\right]$$

$$= \frac{3}{8}\cdot\frac{\pi}{6}\cdot\left[(0+0) + 3(0.5+0.866+0.866+0.5) + 2(1)\right].$$

$$= \frac{\pi}{16}\left[2 + 8.196\right] = \frac{22}{16\times 7}(10.196) = 2.0027.$$

$$\therefore \int_0^{\pi} \sin x \, dx = 2.0027.$$

Note :- The volume generated by revolving the area bounded by the curve $y = f(x)$, the x-axis and the ordinates $x = a$ and $x = b$ is given by $\int_a^b \pi y^2 \, dx$.

1) A curve passes through the points (1, 0.2) (2, 0.7) (3, 1) (4, 1.3) (5, 1.5) (6, 1.7) (7, 1.9) (8, 2.1) (9, 2.3). Using (i) Trapezoidal rule (ii) Simpson's $\frac{1}{3}$rd rule estimate the volume generated by revolving the area between the curve, the x-axis and the ordinates $x = 1$ and $x = 9$ about the x-axis.

Sol:- The volume generated by revolving the area bounded by the curve $y = f(x)$ the x-axis and the ordinates $x = a$ and $x = b$ is $\int_{x=a}^{x=b} \pi y^2 dx$.

Here $a = 1$, $b = 9$. $h = \frac{b-a}{n} = \frac{9-1}{8} = 1$

$y_0 = 0.2 \quad y_0^2 = 0.04 \qquad y_1 = 0.7 \quad y_1^2 = 0.49$

$y_2 = 1 \quad y_2^2 = 1 \qquad y_3 = 1.3 \quad y_3^2 = 1.69$ 16

$y_4 = 1.5 \quad y_4^2 = 2.25 \qquad y_5 = 1.7 \quad y_5^2 = 2.89$

$y_6 = 1.9 \quad y_6^2 = 3.61 \qquad y_7 = 2.1 \quad y_7^2 = 4.41$

$y_8 = 2.3 \quad y_8^2 = 5.29.$

(i) Trapezoidal rule :-

Required volume of the solid $= \pi \int_{x=1}^{x=9} y^2\,dx.$

$$= \pi \frac{h}{2}\left[(y_0^2 + y_8^2) + 2(y_1^2 + y_2^2 + \cdots + y_7^2)\right]$$

$$= \frac{\pi}{2}\left[(0.04 + 5.29) + 2(0.49 + 1 + 1.69 + 2.25 + 2.89 + 3.61 + 4.4)\right]$$

$$= \frac{22}{7\times 2}\left[5.33 + 32.66\right] = \frac{22}{14}\left[37.99\right] = 59.699$$

$\therefore$ volume of the solid $= 59.699$ cubic units.

(ii) Simpson's $\frac{1}{3}$rd rule :—

Required volume of the solid $= \pi \int_{x=1}^{x=9} y^2\,dx.$

$$= \pi \cdot \frac{h}{3}\left[(y_0^2 + y_8^2) + 4(y_1^2 + y_3^2 + y_5^2 + y_7^2) + 2(y_2^2 + y_4^2 + y_6^2)\right]$$

$$= \pi \cdot \frac{1}{3}\left[(0.04 + 5.29) + 4(0.49 + 1.69 + 2.89 + 4.41) + 2(1 + 2.25 + 3.61)\right].$$

$$= \frac{22}{21}\left[5.33 + 37.92 + 13.72\right] = 59.68.$$

$\therefore$ volume of the solid $= 59.68$ cubic units.

The velocity v of a particle at a distance s from a point on its path is given by the following table. (17)

s (ft)	0	10	20	30	40	50	60
v (ft/s)	47	58	64	65	61	52	38

Estimate the time taken to travel 60 ft using (i) Simpson's $\frac{1}{3}$rd rule (ii) Simpson's $\frac{3}{8}$th rule. (iii) Trapezoidal rule.

Sol:- We know that the velocity is the rate of change of displacement

i.e $\frac{ds}{dt} = v$.

The time taken to travel 60 ft is given by

$$dt = \frac{ds}{v}$$

$$t = \int dt = \int_0^{60} \frac{1}{v}\, ds$$

$$t = \int_0^{60} \frac{1}{v}\, ds$$

Take $s = x$ and $y = \frac{1}{v}$.

$$t = \int_0^{60} y\, dx$$

$y_0 = \frac{1}{v_0} = \frac{1}{47} = 0.02128$ $\qquad$ $y_1 = \frac{1}{v_1} = \frac{1}{58} = 0.01724$

$y_2 = \frac{1}{v_2} = \frac{1}{64} = 0.015625$ $\qquad$ $y_3 = \frac{1}{v_3} = \frac{1}{65} = 0.01538$

$y_4 = \frac{1}{v_4} = \frac{1}{61} = 0.016393$ $\qquad$ $y_5 = \frac{1}{v_5} = \frac{1}{52} = 0.019231$

$y_6 = \frac{1}{v_6} = \frac{1}{38} = 0.026316$.

Here $h = 10$. $\left[\because h = \frac{b-a}{n}\right.$ $\left. h = \frac{60-0}{6} = 10\right]$

Simpson's $\frac{1}{3}$rd rule :-

$$t = \int_0^{60} y\,dx = \frac{h}{3}\left[(y_0+y_6) + 4(y_1+y_3+y_5) + 2(y_2+y_4)\right]$$

$$= \frac{10}{3}\left[(0.02128 + 0.026316) + 4(0.01724 + 0.01538 + 0.019231) + 2(0.015625 + 0.016393)\right].$$

$$= \frac{10}{3}\left[0.047596 + 0.207404 + 0.064036\right]$$

$$t = 1.06345$$

(18)

(ii) Simpson's $\frac{3}{8}$th rule :—

$$t = \int_0^{60} y\,dx = \frac{3h}{8}\left[(y_0+y_6) + 3(y_1+y_2+y_4+y_5) + 2(y_3)\right]$$

$$= \frac{3(10)}{8}\left[(0.02128 + 0.026316) + 3(0.01724 + 0.015625 + 0.016393 + 0.019231) + 2(0.01538)\right]$$

$$= \frac{30}{8}\left[0.047596 + 0.205467 + 0.03076\right]$$

$$t = 1.0643 .$$

(1) Evaluate $\int_0^1 \frac{dx}{1+x^2}$ using Trapezoidal rule by taking $h=0.5$ $h=0.25$ and $h=0.125$. Ans: 0.775, 0.7828, 0.78415.

(2) A rocket is launched from the ground. Its acceleration measured every 5 seconds is tabulated below. Find the velocity and position of the rocket at $t = 40$ seconds. Use Trapezoidal rule as well as Simpson's rule.

t	0	5	10	15	20	25	30	35	40
a	40	45.25	48.5	51.25	54.35	59.48	61.5	64.3	68.7

Ans: Trapezoidal rule — 2194.9, 87796.

Simpson's rule — 2197.5, 87900.

(3) Evaluate $\int_0^1 e^{-x^2} dx$ by dividing the range of integration into 4 equal parts using (a) Trapezoidal rule (b) Simpson's $\frac{1}{3}$rd rule.

Ans:- 0.7428, 0.7467

(4) Find the area under the curve represented by the following table bounded by x-axis and the ordinates 0.6 and 1.2

x	0.6	0.8	1.0	1.2
y	1.23	1.58	2.03	4.32.

Ans: 1.277.

(5) Evaluate $\int_0^{\pi} \sin x \, dx$ by dividing the range into 10 equal parts using (a) Trapezoidal rule (b) Simpson's $\frac{1}{3}$rd rule. Ans:- 1.9835, 2.0007.

(6) Evaluate $\int_0^{\pi/2} \sqrt{\cos\theta}\, d\theta$ taking $h = \frac{\pi}{12}$ using (a) Trapezoidal rule. (b) Simpson's $\frac{1}{3}$rd rule Ans:- 1.1702, 1.18718.

(7) Find $\int_0^1 \frac{dx}{1+x}$ using 10 intervals using simpson's rule

2.

Ans:- 0.6931689.

(8) Evaluate $\int_0^1 \frac{dx}{1+x^2}$ by taking $h=\frac{1}{6}$ using (i) simpson's $\frac{1}{3}$rd rule.

$h=\frac{1}{6}$ using (ii) simpson's $\frac{3}{8}$th rule.

Ans:- 0.785388, 0.785398

(9) The velocities of a car at intervals of 2 minutes are given below.

Time in minutes	0	2	4	6	8	10	12
Velocity in Km/hr	0	22	30	27	18	7	0

Find the distance covered by the car.

(i) Using simpson's $\frac{1}{3}$rd rule (ii) Using simpson's $\frac{3}{8}$th rule.

Ans: 3.5555, 3.5625.

(10) The velocity v of a particle at a distance s from a point on its path is given by the following table.

s (ft)	0	10	20	30	40	50	60
v (ft/s)	47	58	64	65	61	52	38

Estimate the time taken to travel 60 ft using.

(i) simpson's $\frac{1}{3}$rd rule (ii) Simpson's $\frac{3}{8}$th rule.

Ans:- 1.063518, 1.0643723.

(11) A curve passes through the points (1, 0.2) (2, 0.7) (3, 1) (4, 1.3) (5, 1.5) (6, 1.7) (7, 1.9) (8, 2.1) (9, 2.3) using simpson's $\frac{1}{3}$rd rule estimate the volume generated by revolving the area between the curve the x-axis and the ordinates $x=1$ and $x=9$ about the x-axis.

Ans:- 18.99.

(12) A curve is drawn to pass through the points given by the following table.

x	1	1.5	2	2.5	3	3.5	4
y	2	2.4	2.7	2.8	3	2.6	2.1

Find the area bounded by the curve the x-axis and the ordinates $x=1$ and $x=4$. Ans :- 7.74. (3)

(13) A solid of revolution is formed by rotating about the x-axis the area b/n x-axis and times $x=0$ and $x=1$ and a curve through the points with the following co ordinate.

x	0	0.25	0.5	0.75	1
y	1	0.9896	0.9589	0.9089	0.8415

Find the volume of the solid formed Ans:- 2.8192

(14) The velocity v of a particle at distance s from a point in its path is given by

S	0	10	20	30	40	50	60
V	47	58	64	65	61	52	38

Estimate the time taken to travel 60ft using simpson's $\frac{1}{3}$rd rule.

Ans:- 1.063.

(15) calculate $\int_0^{\pi/2} e^{\sin x}\,dx$ using simpson's rule by taking 7 ordinates and correct to 4 decimal places Ans:- 3.1043.

(16) Evaluate $\int_0^1 \frac{dx}{1+x^2}$ using simpson's $\frac{1}{3}$rd, simpson's $\frac{3}{8}$th and Trapezoidal rule - Hence obtain the approximate value of π in each. Compare the result with exact value.

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(17) Find $\int_0^1 \frac{x^2}{1+x^3}\,dx$. using simpson's $\frac{1}{3}$rd rule by taking 4 Subintervals.

Also find the error. Ans:- 0.231066, 0.000026. (4)

(18) The velocity of a train which starts from rest is given by the following table.

t min	2	4	6	8	10	12	14	16	18	20
V km/hr	16	28.8	40	46.4	51.2	32	17.6	8	3.2	0

Estimate total distance run in 20 min. Ans:- 8.25 km.

(19) Find the value of $\int_1^5 \log_{10} x\,dx$ taking 8 sub intervals correct to 4 significant figures by Trapezoidal and simpson's $\frac{1}{3}$rd rule.

Ans. 1.7505025.

(20) The table below shows the velocities of a moped which starts from rest at fixed intervals of time. Find the distance traveled by the moped in 20 min.

Time (t)	2	4	6	8	10	12	14	16	18	20
Velocity (v)	0	10	18	25	29	32	20	11	5	2

Ans:- 309.33 Km.

(21) Evaluate $\int_{0.2}^{1.4} (\sin x - \log_e^x + e^x)\,dx$ by dividing the interval into (i) 6 equal parts

(ii) 12 equal parts Ans:- 4.05116.

(22) A solid of revolution is formed by rotating about the x-axis, the area between the x-axis the lines $x=0$ and $x=1$ and a curve through the points with the following co ordinates.

x	0	0.25	0.5	0.75	1
y	1	0.9896	0.9589	0.9089	0.8415

Estimate the volume of the solid formed using simpson's rule.
Ans:- 2.8192.

23) The velocity v of a particle at distance s from a point on its linear path is given by the following table. (5)

s (m)	0	2.5	5	7.5	10	12.5	15	17.5	20
v (m/sec)	16	19	21	22	20	17	13	11	9

Estimate the time taken by the particle to traverse the distance of 20 metres using simpson's rule.

24) The velocity v of a particle at distances s from a point on its path given by the table.

s ft	0	10	20	30	40	50	60
v ft/sec	47	58	64	65	61	52	38.

Estimate the time taken to travel 60 ft by using simpson's $\frac{1}{3}$ rule. Compare the result with simpson's $\frac{3}{8}$ rule.

25) The following table gives the velocity v of a particle at time t.

t (sec)	0	2	4	6	8	10	12
v (m/sec)	4	6	16	34	60	94	136

Find the distance moved by the particle in 12 seconds and also the acceleration at t = 2 sec.

26) A rocket is launched from the ground. Its acceleration is registered during the first 80 seconds and is given in the table below. Using simpson's $\frac{1}{3}$rd rule, find the velocity of the rocket at t = 80 sec.

t (sec)	0	10	20	30	40	50	60	70	80
f (cm/sec²)	30	31.63	33.34	35.47	37.75	40.33	43.25	46.69	50.67

27) A curve is drawn to pass through the points given by the following table.

x	1	1.5	2	2.5	3	3.5	4
y	2	2.4	2.7	2.8	3	2.6	2.1

Estimate the area bounded by the curve, x-axis and the lines $x=1$, $x=4$

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28) A river is 80 ft wide. The depth d in feet at a distance x ft from one bank is given by the following table.

x	0	10	20	30	40	50	60	70	80
y	0	4	7	9	12	15	14	8	3.

Find approximately the area of the cross section

A rocket is launched from the ground Its accelera

29) A reservoir discharging water through sluices at a depth h below the water surface has a surface area A for various values of h as given below.

h (ft)	10	11	12	13	14
A (sq.ft)	950	1070	1200	1350	1530.

If t denotes time in minutes, the rate of fall of the surface is given by $\frac{dh}{dt} = -48\frac{\sqrt{h}}{A}$ Estimate the time taken for the water level to fall from 14 to 10 ft above the sluices.

30) Evaluate (a) $\int_1^2 \cos x \, dx$ using the trapezoidal rule with $h = \frac{1}{2}$. Compare with the exact solution.

(b) $\int_0^{\pi/2} e^{-x} \cos x \, dx$ using the simpson's $\frac{1}{3}$rd rule with $h = \frac{\pi}{8}$.

R.NO	Q.NO
1-20	1-10
21-40	11-20
41-60	21-30